

Quasi-Töplitz Functions in KAM Theorem

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Abstract

We define and describe the class of Quasi-Töplitz functions. We then prove an abstract KAM theorem where the perturbation is in this class. We apply this theorem to a Non-Linear-Schrödinger equation on the torus \mathbb{T}^d , thus proving existence and stability of quasi-periodic solutions and recovering the results of [10]. With respect to that paper we consider only the NLS which preserves the total Momentum and exploit this conserved quantity in order to simplify our treatment.

1 Introduction

In this paper, we study a model NLS with external parameters on the torus \mathbb{T}^d . The purpose is to apply KAM theory and prove existence and stability of quasi-periodic solutions. We focus on the equation

$$iu_t + \Delta u + M_\xi u + f(|u|^2)u = 0, \quad x \in \mathbb{T}^d, \quad t \in \mathbb{R}, \quad (1.1)$$

where $f(y)$ is a real analytic function with $f(0) = 0$, while M_ξ is a Fourier multiplier, namely a linear operator which commutes with the Laplacian and whose role is to introduce b parameters in order to guarantee that equation (1.1) linearized at $u = 0$ admits a quasi-periodic solution with b frequencies. More precisely, let $\phi_n(x) = \sqrt{\frac{1}{4\pi^2}} e^{i\langle n, x \rangle}$, $n \in \mathbb{Z}^d$ be the standard Fourier basis, we choose a finite set $\{\mathbf{n}^{(1)}, \dots, \mathbf{n}^{(b)}\}$ with $\mathbf{n}^{(i)} \in \mathbb{Z}^d$ and define M_ξ so that the eigenvalues of the operator $\Delta + M_\xi$ are

$$\begin{cases} \omega_j &= |\mathbf{n}^{(j)}|^2 + \xi_j, & 1 \leq j \leq b \\ \Omega_n &= |n|^2, & n \notin \{\mathbf{n}^{(1)}, \dots, \mathbf{n}^{(b)}\} \end{cases} \quad (1.2)$$

Key words: Schrödinger equation, KAM Tori, Quasi Töplitz

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Equation (1.1) is a well known model for the natural NLS, in which the Fourier multiplier is substituted by a multiplicative potential V . Existence and stability of quasi-periodic solutions via a KAM algorithm was proved in [10] for the more general case where $f(y)$ is substituted with $f(y, x)$, $x \in \mathbb{T}^d$. With respect to that paper we strongly exploit the fact that equation has the total momentum $M = \int_{\mathbb{T}^d} \bar{u} \nabla u$ as an integral of motion, this induces some significant simplifications which we think are interesting. Our dynamic result is

Theorem 1 *There exists a positive-measure Cantor set \mathcal{C} such that for any $\xi = (\xi_1, \dots, \xi_b) \in \mathcal{C}$, the above nonlinear Schrödinger equation (1.1) admits a linearly stable small-amplitude quasi-periodic solution.*

Before giving a more detailed comparison let us make a brief excursus on the literature on quasi-periodic solutions for PDEs on \mathbb{T}^d and on the general strategy of a KAM algorithm.

The existence of quasi-periodic solutions for equation (1.1) (as well as for the non-linear wave equation) was first proved by Bourgain, see [3] and [4], by applying a combination of Lyapunov-Schmidt reduction and Nash-Moser generalized implicit function theorem, in order to solve the small divisor problem. This method is very flexible and may be effectively applied in various contexts, for instance in the case where $f(y)$ has only finite regularity, see [6] and [7]. As a drawback this method only establishes existence of the solutions but not the existence of a stable normal form close to them. In order to achieve this stronger result it is natural to extend to (1.1) on \mathbb{T}^d by now classical KAM techniques, which were developed to study equation (1.1) with Dirichlet boundary conditions on the segment $[0, \pi]$. A fundamental hypothesis in the aforementioned algorithms is that the eigenvalues Ω_n are simple, and this is clearly not satisfied already in the case of equation (1.1) on \mathbb{T}^1 , where the eigenvalues are double. We mention that this hypothesis was weakened for the non-linear wave equation in [8] by only requiring that the eigenvalues have finite and uniformly bounded multiplicity. Their method however does not extend trivially to the NLS on \mathbb{T}^1 and surely may not be applied to the NLS in higher dimension, where the multiplicity of Ω_n is of order $\Omega_n^{(d-1)/2}$. The first result on KAM theory on the torus \mathbb{T}^d was given in [12] for the *non-local* NLS:

$$iu_t + \Delta u + M_\xi u + f(|\Psi_s(u)|^2)\Psi_s(u) = 0, \quad x \in \mathbb{T}^d, \quad t \in \mathbb{R},$$

where Ψ_s is a linear operator, diagonal in the Fourier basis and such that $\Psi_s(\phi_n) = |n|^{-2s}\phi_n$ for some $s > 0$. The key points of that paper are: 1.

the use of the conservation of the total momentum to avoid the problems arising from the multiplicity of the Ω_n and 2. the fact that the presence of the non-local operator Ψ_s simplifies the proof of the *Melnikov non-resonance conditions* throughout the KAM algorithm.

Let us briefly describe the general strategy in the KAM algorithm for equation (1.1).

We expand the solution in Fourier series as $u = \sum_{n \in \mathbb{Z}^d} q_n \phi_n(x)$ and introduce standard action-angle coordinate: $q_{\mathbf{n}_j} = \sqrt{I_j} e^{i\theta_j}$, $j = 1, \dots, b$; $q_n = z_n$, $n \neq \{\mathbf{n}^{(1)}, \dots, \mathbf{n}^{(b)}\}$. We get

$$H = \sum_{1 \leq j \leq b} \omega_j I_j + \sum_{n \in \mathbb{Z}_1^d} \Omega_n |z_n|^2 + P(I, \theta, z, \bar{z}), \quad \mathbb{Z}_1^d := \mathbb{Z}^d \setminus \{\mathbf{n}_1, \dots, \mathbf{n}_b\}. \quad (1.3)$$

It is easily seen that H and hence P preserve the total momentum (see 2.5 below) moreover P (and $\sum (\Omega_m - |m|^2) z_m \bar{z}_m$) are Töplitz/anti-Töplitz functions, namely the Hessian matrix $\partial_{z_m} \partial_{\bar{z}_n} P = H^{\sigma, \sigma'}(\sigma m + \sigma' n, z, \bar{z})$.

Informally speaking the KAM algorithm consists in constructing a convergent sequence of symplectic transformations such that

$$\Phi_\nu H = H_\nu = \sum_{1 \leq j \leq b} \omega_j^{(\nu)}(\xi) I_j + \sum_{n \in \mathbb{Z}_1^d} \Omega_n^{(\nu)}(\xi) |q_n|^2 + P_\nu(\xi, I, \theta, z, \bar{z}), \quad (1.4)$$

where $P_\nu \rightarrow 0$ in some appropriate norm. The symplectic transformation is well defined for all ξ which satisfy the *Melnikov non-resonance conditions*:

$$|\langle \omega^{(\nu')}, k \rangle + \Omega^{(\nu')} \cdot l| \geq \gamma_{\nu'} (1 + |k|)^{-\tau}, \quad (1.5)$$

for all $\nu' \leq \nu$ and $\forall k \in \mathbb{Z}^b$, $l \in \mathbb{Z}_1^d$ such that $(k, l) \neq (0, 0)$, $|l| \leq 2$. Here τ is an appropriate constant, while γ_ν is a sequence of positive numbers. With this conditions in mind it is clear that a degeneracy $\Omega_n^{(\nu')} = \Omega_m^{(\nu')}$ poses problems since the left hand side in (1.5) is identically zero for $h = 0, l = e_m - e_n$ (e_m with $m \in \mathbb{Z}_1^d$ is the standard basis vector). To avoid this problem we use the fact that all the H_ν have M as constant of motion. This in turn implies that some of the Fourier coefficients of P_ν are identically zero so that the conditions (1.5) need to be imposed only on those k, l such that $\sum_{i=1}^b \mathbf{n}_i k_i + \sum_{m \in \mathbb{Z}_1^d} m l_m = 0$. Then, in our example, $k = 0$ automatically implies $n = m$. This is the key argument used in [12]. However, once that one has proved that the left hand side of (1.5) is never identically zero, one still has to show that the quantitative bounds of (1.5) may be imposed on some positive measure set of parameters ξ . This is an easy task when

$|l| = 0, 1$ or $l = e_m + e_n$ but may pose serious problems in the case $l = e_m - e_n$ where the non-resonance condition is

$$|\langle \omega^{(\nu')}, k \rangle + \Omega_m^{(\nu')} - \Omega_n^{(\nu')}| \geq \gamma_{\nu'}(1 + |k|)^{-\tau}, \quad \forall k \in \mathbb{Z}^b, \quad n, m \in \mathbb{Z}_1^d \quad (1.6)$$

where $n - m = \sum_{i=1}^b \mathbf{n}_i k_i$. Indeed in this case for every fixed value of k one should in principle impose infinitely many conditions, since the momentum conservation only fixes $n - m$. In [12], the presence of Ψ_s implies that $\Omega_m^{(\nu)} - |m|^2 \approx \frac{\varepsilon}{|m|^s}$ so that if $|m|^s > c|k|^\tau$ the variation of $\Omega^{(\nu)}$ is negligible. This implies in turn that one has to impose only finitely many conditions for each k . In the case of equation (1.1) however $s = 0$, so that this argument may not be applied. To show that it is still true that one may impose the non resonance conditions by verifying only a finite number of bounds for each k one needs some control on $\Omega_m^{(\nu)} - |m|^2$, for $|m|$ large, throughout the KAM algorithm. The ideal setting is when $\Omega_m^{(\nu)} - |m|^2$ is m -independent. This holds true for the first step of the KAM algorithm due to the fact that P is a Töplitz function. However it is easily seen that already P_1 is not a Töplitz function. Our strategy is to define a class of functions, the *quasi-Töplitz functions*, and show that all the P_ν belong to this set. Informally speaking a quasi-Töplitz function is a function whose Hessian restricted to affine subspaces defined by equations with integer coefficients is *well approximated* by a Töplitz matrix.

To explain the use of this property we give a more detailed description of the control that we have on the variation of the normal frequencies. Let τ_0 be a parameter such that

$$|\langle \omega, k \rangle - h| > \gamma_0(1 + |k|)^{-\tau_0}, \quad (k, h) \neq (0, 0)$$

may hold for a positive measure set of ξ . We introduce a parameter $\tau > \tau_0$, fixed in formula (2.6).

The correction to the normal frequency $\Omega_m^{(\nu-1)}$ is given (at step ν) by the diagonal terms in $\langle \partial_{z_m} \partial_{\bar{z}_m} P_{\nu-1} |_{z=\bar{z}=0} \rangle$. The quasi-Töplitz property (see Definition 2.6 and Formula (4.10)) states that for all K large enough ($K \geq K_\nu$, the ultraviolet cut-off at step ν) and for all $|m| > K^\tau$ there exists a parameter $\tau_0 < \tau_1 < \tau/4d$ such that

$$\Omega_m = |m|^2 + \tilde{\Omega}[m] + \bar{\Omega}_m K^{-4d\tau_1},$$

where the function $\tilde{\Omega}$ assumes at most $K^{3d\tau_1}$ different values, while $\bar{\Omega}_m$ is bounded. The value of $\tilde{\Omega}$ on m is determined by the notion of *standard cut*, see Definition 2.3 and Lemma 2.2, which assigns to each point $|m| > K^\tau$ a

portion of an affine subspace (depending on K) on which $\tilde{\Omega}$ is constant. We show that such subspaces are finite (bounded by $K^{3d\tau_1}$). Finally we show that one may verify all the conditions (1.6) by imposing them only for one point for each affine subspace.

Eliasson and Kuksin in [10] consider an NLS equation which does not have M as a constant of motion. This implies that some of the Melnikov non-resonance conditions (1.6) may not be imposed, and terms like $P_{0,0,n,m}$ with $|n| = |m|$ are kept, at each step of the KAM algorithm they thus obtain a more complicated normal form, which then must be reduced to the standard one through a linear symplectic change of variables.

To deal with the variation of the normal frequencies they introduce a property, Töplitz-Lipschitz, which can be preserved by KAM iteration. In the definition of Töplitz-Lipschitz, if M is the Hessian matrix corresponding to an analytic function, they require that for any $a, b, c \in \mathbb{Z}^d$, the limit $M(\pm, c) := \lim_{t \rightarrow \infty} M_{b+tc}^{a \pm tc}$ exists; and the limit matrix $M(\pm, c)$ still required to satisfy this condition in a lower dimension space \mathbb{Z}^{d-1} . The speed tends to the limit is controlled by $\frac{1}{t}$. Whats more, except a finite set, they cover any neighborhood $\{|a - b| \leq N\} \subset \mathbb{Z}^d \times \mathbb{Z}^d$ of diagonal by finite Lipschitz domain. This reduces the infinitely many second Melnikov conditions to a finite number. In [13] an understanding of this property in \mathbb{T}^2 is given.

2 Relevant notations and definitions

2.1 Function spaces and norms

We start by introducing some notations. We fix b vectors $\{\mathbf{n}^{(1)}, \dots, \mathbf{n}^{(b)}\}$ in \mathbb{Z}^d called the *tangential sites*. We denote by $\mathbb{Z}_1^d := \mathbb{Z}^d \setminus \{\mathbf{n}^{(1)}, \dots, \mathbf{n}^{(b)}\}$ the complement, called the *normal sites*. Let $z = (\dots, z_n, \dots)_{n \in \mathbb{Z}_1^d}$, and its complex conjugate $\bar{z} = (\dots, \bar{z}_n, \dots)_{n \in \mathbb{Z}_1^d}$. We introduce the weighted norm

$$\|z\|_\rho = \sum_{n \in \mathbb{Z}_1^d} |z_n|^2 e^{2|n|\rho} |n|^{d+1},$$

where $|n| = \sqrt{n_1^2 + n_2^2 + \dots + n_d^2}$, $n = (n_1, n_2, \dots, n_d)$ and $\rho > 0$. We denote by ℓ_ρ the Hilbert space of lists $\{w_j = (z_j, \bar{z}_j)\}_{j \in \mathbb{Z}_1^d}$ with $\|z\|_\rho < \infty$.

We consider the real torus $\mathbb{T}^b := \mathbb{R}^b / \mathbb{Z}^b$ as naturally contained in the space $\mathbb{C}^b / \mathbb{Z}^b \times \ell_\rho$ as the subset where $I = z = \bar{z} = 0$. We then consider in this space the neighborhood of \mathbb{T}^b :

$$D(r, s) := \{(\theta, I, z, \bar{z}) : |\operatorname{Im} \theta| < s, |I| < r^2, \|z\|_\rho < r, \|\bar{z}\|_\rho < r\},$$

where $|\cdot|$ denotes the sup-norm of complex vectors. Denote by \mathcal{O} an open and bounded parameter set in \mathbb{R}^b and let $D = \max_{\xi, \eta \in \mathcal{O}} |\xi - \eta|$.

We consider functions $F(I, \theta, z; \xi) : D(r, s) \times \mathcal{O} \rightarrow \mathbb{C}$ analytic in I, θ, z and of class C_W^1 in ξ . We expand in Taylor–Fourier series as:

$$F(\theta, I, z, \bar{z}; \xi) = \sum_{l, k, \alpha, \beta} F_{lk\alpha\beta}(\xi) I^l e^{i\langle k, \theta \rangle} z^\alpha \bar{z}^\beta, \quad (2.1)$$

where the coefficients $F_{lk\alpha\beta}(\xi)$ are of class C_W^1 (in the sense of Whitney), the vectors $\alpha \equiv (\cdots, \alpha_n, \cdots)_{n \in \mathbb{Z}_1^d}$, $\beta \equiv (\cdots, \beta_n, \cdots)_{n \in \mathbb{Z}_1^d}$ have finitely many non-zero components $\alpha_n, \beta_n \in \mathbb{N}$, $z^\alpha \bar{z}^\beta$ denotes $\prod_n z_n^{\alpha_n} \bar{z}_n^{\beta_n}$ and finally $\langle \cdot, \cdot \rangle$ is the standard inner product in \mathbb{C}^b .

If \mathcal{S} is a set of monomials in $I_j, e^{i\theta_j}, z_m, \bar{z}_n$, we define the projection operator $\Pi_{\mathcal{S}}$ which to a given analytic function F associates the part of the series only relative to the monomials in \mathcal{S} .

The analyticity of the functions implies total convergence of the Taylor–Fourier series with respect to the following weighted norm of F :

$$\|F\|_{r,s} = \|F\|_{D(r,s), \mathcal{O}} \equiv \sup_{\substack{\|z\|_\rho < r \\ \|\bar{z}\|_\rho < r}} \sum_{\alpha, \beta, k, l} |F_{kl\alpha\beta}|_{\mathcal{O}} r^{2|l|} e^{|k|s} |z^\alpha| |\bar{z}^\beta|, \quad (2.2)$$

$$|F_{kl\alpha\beta}|_{\mathcal{O}} \equiv \sup_{\xi \in \mathcal{O}} (|F_{kl\alpha\beta}| + |\frac{\partial F_{kl\alpha\beta}}{\partial \xi}|). \quad (2.3)$$

(the derivatives with respect to ξ are in the sense of Whitney). To an analytic function F , we associate a Hamiltonian vector field with coordinates

$$X_F = (F_I, -F_\theta, \{iF_{z_n}\}_{n \in \mathbb{Z}_1^d}, \{-iF_{\bar{z}_n}\}_{n \in \mathbb{Z}_1^d}).$$

We say that F is regular if the function $(I, \theta, z, \bar{z}) \rightarrow X_F$ is analytic from $D(r, s) \rightarrow \mathbb{C}^{2b} \times \ell_\rho$. Its weighted norm is defined by ¹

$$\begin{aligned} \|X_F\|_{r,s} = \|X_F\|_{D(r,s), \mathcal{O}_0} &\equiv \|F_I\|_{D(r,s), \mathcal{O}_0} + \frac{1}{r^2} \|F_\theta\|_{D(r,s), \mathcal{O}_0} \\ &+ \frac{1}{r} \|(\partial_z F, \partial_{\bar{z}} F)\|_{r,s}. \end{aligned} \quad (2.4)$$

A function F is said to satisfy momentum conservation if $\{F, M\} = 0$ with $M = \sum_{i=1}^b \mathbf{n}^{(i)} I_i + \sum_{j \in \mathbb{Z}_1^d} j |z_j|^2$. This implies that

$$F_{k,l,\alpha,\beta} = 0, \quad \text{if } \pi(k, \alpha, \beta) := \sum_{i=1}^b \mathbf{n}^{(i)} k_i + \sum_{j \in \mathbb{Z}_1^d} j(\alpha_j - \beta_j) \neq 0. \quad (2.5)$$

¹The norm $\|\cdot\|_{D(r,s), \mathcal{O}}$ for scalar functions is defined in (2.2). The vector function $G : D(r, s) \times \mathcal{O} \rightarrow \mathbb{C}^m$, ($m < \infty$) is similarly defined as $\|G\|_{D(r,s), \mathcal{O}} = \sum_{i=1}^m \|G_i\|_{D(r,s), \mathcal{O}}$.

By Jacobi's identity momentum conservation is preserved by Poisson bracket.

Definition 2.1 We denote by $\mathcal{A}_{r,s}$ the space of regular analytic functions in $D(r,s)$ and C_W^1 in \mathcal{O} which satisfy momentum conservation (2.5).

We have following useful result

Lemma 2.1 i) $\|\Pi_S f\|_{r,s} \leq \|f\|_{r,s}$.

ii) $\mathcal{A}_{r,s}$ is closed through Poisson brackets, with respect to the symplectic form $dI \wedge d\theta + idz \wedge d\bar{z}$, moreover by Cauchy estimates, if we denote $\delta = \min(1 - \frac{r'}{r}, s - s')$,

$$\|\{f, g\}\|_{r',s'} \leq \delta^{-1} \|X_f\|_{r',s'} \|g\|_{r',s'}, \quad \|[X_f, X_g]\|_{r',s'} \leq 2\delta^{-1} \|X_f\|_{r,s} \|X_g\|_{r,s}.$$

The first result is obvious and the proof of the second please refer to Geng-You [11]

In the course of our analysis we shall need to fix several constants (which will be determined by the KAM algorithm). We start by fixing some large numbers

$$\tau_0 > \max(d + b, 12), \quad \tau := (4d)^{d+1}(\tau_0 + 1). \quad (2.6)$$

2.2 Affine subspaces

2.2.1 Optimality

An affine space A of codimension ℓ in \mathbb{R}^d can be defined by a list of ℓ equations $A := \{x \mid v_i \cdot x = p_i\}$ where the v_i are independent row vectors in \mathbb{R}^d . We will denote $A = [v_i; p_i]_\ell$. We will be interested in particular in the case when v_i, p_i have integer coordinates, i.e. are *integer vectors*.

We denote by

$$\langle v_i \rangle_\ell = \text{Span}(v_1, \dots, v_\ell; \mathbb{R}) \cap \mathbb{Z}_1^d, \quad B_K^a := \{x \in \mathbb{R}^d \mid |x| < C_1 K\} \cap \mathbb{Z}_1^d \setminus \{0\},$$

here K is any large number and the constant C_1 depends only on the tangential sites \mathbf{n}_i .

In the set of vectors \mathbb{Z}^m we can define the *sign lexicographical order* as follows. Given $a = (a_1, \dots, a_m)$ set $(|a|) := (|a_1|, \dots, |a_m|)$ then we set $a \prec b$ if either $(|a|) < (|b|)$ in the lexicographical order (over \mathbb{N}) or if $(|a|) = (|b|)$ and $a < b$ in the lexicographical order in \mathbb{Z} . With this definition every non empty set of elements in \mathbb{Z}^m has a unique minimum.

In particular consider a fixed but large enough K , and restrict to the set \mathcal{H}_K of all affine spaces A which can be presented as $A = [v_i; p_i]_\ell$ for some

$0 < \ell \leq d$ so that that $v_i \in B_K^a$. We display as $(p_1, \dots, p_\ell; v_1, \dots, v_\ell)$ a given presentation, so that it is a vector in $\mathbb{Z}^{\ell(d+1)}$. Then we can say that $[v_i; p_i]_\ell \prec [w_i; q_i]_\ell$ if $(p_1, \dots, p_\ell; v_1, \dots, v_\ell) \prec (q_1, \dots, q_\ell; v_1, \dots, v_\ell)$

Definition 2.2 *The K -optimal presentation $[l_i; q_i]_\ell$ of A is the minimum in the sign lexicographical order of the presentations of A which satisfy the previous bounds.*

Remark 2.1 *While it is possible that an affine subspace is not in \mathcal{H}_K , so that it does not have a K optimal presentation; each point m does have a K -optimal presentation.*

Lemma 2.1 *If the presentation $A = [v_i; p_i]_\ell$ is K -optimal, we have for all $j < \ell$ and for all $v \in B_K^a \setminus \langle v_1, \dots, v_j \rangle$:*

$$|p_1| \leq |p_2| \leq \dots \leq |p_j|, \quad |(v, r)| \geq |p_{j+1}|. \quad (2.7)$$

Proof: If we permute the vectors v_j we have a different presentation, thus the new vector with the permuted (p_1, \dots, p_k) is lexicographically higher which implies the first inequality. As for the second, given $v \in B_K^a \setminus \langle v_1, \dots, v_j \rangle$ we can substitute one of the v_h , $h > j$, with ℓ obtaining a new presentation. Again we deduce by minimality in the lexicographical order, that $|(v, x)| \geq |p_h| \geq |p_j|$. \blacksquare

Given an affine subspace $A := \{x | v_i \cdot x = p_i, \quad i = 1, \dots, \ell\}$ by the notation $A \xrightarrow{K} [v_i; p_i]_\ell$ we mean that the given presentation is K optimal.

Remark 2.2 *For fixed K, ℓ, p the number of affine spaces in \mathcal{H}_K of codimension ℓ and such that $|p_\ell| = p$ is bounded by $(CK)^{\ell d} (2p)^{\ell-1}$.*

2.2.2 Cuts

We are particularly interested in K -optimal presentations of points. Given a point m we write $m \xrightarrow{K} [v_i; p_i]$ dropping the index ℓ which for a point is always $\ell = d$. We need to analyze certain *cuts*, by this we mean a pair (ℓ, τ_1) where $0 \leq \ell \leq d$ and τ_1 a suitable positive number. Recall $\tau = (4d)^{(d+1)}(\tau_0 + 1)$, $\tau_0 > \max(d + b, 12)$. Set by convention $p_0 = 0$ and $p_{d+1} = \infty$.

Definition 2.3 *We say that, a pair (ℓ, τ_1) where $0 \leq \ell \leq d$ and $\tau_0 \leq \tau_1 \leq \tau/4d$ is a compatible cut for the point $m \xrightarrow{K} [v_i; p_i]$, with parameters $\frac{1}{2} \leq \lambda, \mu \leq 4$, if ℓ is such that $|p_\ell| < \mu K^{\tau_1}$, $|p_{\ell+1}| > \lambda K^{4d\tau_1}$.*

Notice that once we have fixed K, τ_1, λ, μ , for any given $m \in \mathbb{Z}_1^d$ there is at most **one** choice of ℓ such that m has a (ℓ, τ_1) cut with parameters λ, μ .

Of course, if (ℓ, τ_1) is a compatible cut for the point $m \xrightarrow{K} [v_i; p_i]$, with parameters $\frac{1}{2} \leq \lambda', \mu' \leq 4$ it is also so for parameters λ, μ with $\lambda \leq \lambda', \mu' \leq \mu$. In particular we are interested in the extremal case where $\mu = 1/2, \lambda = 4$, in this case we say that (ℓ, τ_1) is a *standard cut* for the point $m \xrightarrow{K} [v_i; p_i]$.

The following construction will be useful: we divide

$$[4K^{4d\tau_0}, \frac{1}{2}K^{\tau/4d}) = \cup_{i=1}^{d-1} [K^{S_i}, K^{S_{i+1}}] \cup [K^{S_d}, \frac{1}{2}K^{\tau/4d})$$

by setting $K^{S_1} = 4K^{4d\tau_0}$ and defining recursively

$$K^{S_{i+1}} = 4 \cdot 2^{4d} K^{4dS_i}, \quad i = 1, \dots, d-1.$$

By definition we get

$$2K^{S_d} = 8^{\sum_{i=0}^{d-1} (4d)^i} K^{(4d)^d \tau_0} \leq K^{d(4d)^{d-1} + (4d)^d \tau_0} \leq K^{\tau/4d}$$

since $K > 8$ and $\tau \geq (4d)^{d+1}(\tau_0 + 1)$.

Lemma 2.2 *Each point $m \in \mathbb{Z}_1^d$ has a standard compatible cut (ℓ, τ_1) for some $0 \leq \ell \leq d$ and $\tau_0 \leq \tau_1 \leq \tau/4d$. If $|m| \geq K^\tau$, then $\ell < d$.*

Proof: Let $m \xrightarrow{K} [v_i; p_i]$. If $|p_d| < \frac{1}{2}K^{\tau/4d}$ then we set $\ell = d$ and $\tau_1 = \tau/4d$. If $|p_1| > 4K^{4d\tau_0}$ then we set $\ell = 0$ and $\tau_1 = \tau_0$. Otherwise if $|p_1| < 4K^{4d\tau_0}$ and $|p_d| > \frac{1}{2}K^{\tau/4d}$ then at least one of the $d-1$ intervals $(K^{S_i}, K^{S_{i+1}})$ with $i = 1, \dots, d-1$ does not contain any element of the ordered list $\{|p_2|, \dots, |p_{d-1}|\}$. The parameters (ℓ, τ_1) are fixed by setting $\frac{1}{2}K^{\tau_1} = K^{S_{\bar{i}}}$ where \bar{i} is the smallest index such that the interval $(K^{S_{\bar{i}}}, K^{S_{\bar{i}+1}})$ does not contain any points of the list $\{|p_2|, \dots, |p_{d-1}|\}$; finally we choose ℓ accordingly so that $|p_\ell| < K^{S_{\bar{i}}} = \frac{1}{2}K^{\tau_1}$ and $|p_{\ell+1}| > K^{S_{\bar{i}+1}} = 4K^{4d\tau_1}$.

If $|p_d| \leq K^{\frac{\tau}{4d}}$, by Cramers rule $|m| = |V^{-1}p| < 4K^{\tau/4d} K^{d-1} < K^\tau$.

Remark 2.3 *The purpose of defining a cut (ℓ, τ_1) is to separate the numbers p_i into small and large. The number τ_1 gives a quantitative meaning to this statement.*

Let $[v_i; p_i]_\ell \in \mathcal{H}_K$ be a K -optimal presentation of an affine space of codimension $1 \leq \ell < d$ with $|p_\ell| \leq 4K^{\frac{\tau}{4d}}$.

Definition 2.4 *The set:*

$$[v_i; p_i]_\ell^g := \quad (2.8)$$

$$\left\{ x \in [v_i, p_i]_\ell \mid |x| > 4K^\tau, |(v, x)| > 4 \max(K^{4d\tau_0}, 2^{4d}|p_\ell|^{4d}), \forall v \in B_K^a \setminus \langle v_i \rangle_\ell \right\}$$

will be called the K -good portion of the subspace $[v_i; p_i]_\ell$.

Remark 2.4 *For all $\frac{1}{2} \leq \lambda, \mu \leq 4$ and for each point $m \in [v_i; p_i]_\ell^g$ there exists τ_1 such that m has $[v_i; p_i]_\ell$ as a (τ_1, ℓ) cut with parameters (λ, μ) . Indeed it is sufficient to choose $\tau_1 = \tau_0$ if $K_0 \leq |p_\ell|$ and $\mu K_1^\tau = |p_\ell|$ otherwise.*

Lemma 2.3 *Consider $m, r \in \mathbb{Z}_1^d$ such that $|m + \alpha r| < 5K^3$ with $\alpha = \pm 1$. Let $m \xrightarrow{K} [v_i; p_i]$ and $r \xrightarrow{K} [w_i; q_i]$. Suppose that (ℓ, τ_1) is a compatible cut for m with the parameters μ', λ' . Then:*

(i) (ℓ, τ_1) is a compatible cut for the point r , for all the parameters $\frac{1}{2} \leq \lambda < \lambda', \mu' < \mu \leq 4$ such that $(\mu - \mu')K^{\tau_1}, (\lambda' - \lambda)K^{4d\tau_1} > 5K^4$. (ii) If these conditions are satisfied we have $\langle w_1, \dots, w_\ell \rangle = \langle v_1, \dots, v_\ell \rangle$.

Proof: (i) Assume that $(\mu - \mu')K^{\tau_1}, (\lambda' - \lambda)K^{4d\tau_1} > 5K^4$. Write $\alpha(v_i, r) = (v_i, m + \alpha r) - p_i$. For $i \leq \ell$:

$$|(v_i, r)| \leq |p_i| + |v_i||m + \alpha r| < \mu' K^{\tau_1} + 5K^4 \leq \mu K^{\tau_1}. \quad (2.9)$$

From Formula (2.7) by the definition of K -optimal, for all $v \in B_K^a \setminus \langle v_1, \dots, v_\ell \rangle$ one has

$$|(v, r)| = |(v, m) - (v, \alpha r + m)| \geq |p_{\ell+1}| - |v||m + \alpha r| \geq \lambda' K^{4d\tau_1} - 5K^4 \geq \lambda K^{4d\tau_1}. \quad (2.10)$$

This proves (i).

(ii) By induction on i we show that $|q_i| < \mu K^{\tau_1}$ and $w_i \in \langle v_1, \dots, v_\ell \rangle$ for all $i \leq \ell$. By (2.10) if $v \in B_K^a \setminus \langle v_1, \dots, v_\ell \rangle$ we have $|(v, m)| \geq |p_{\ell+1}| > \lambda K^{4d\tau_1}$.

For $0 \leq i < \ell$, suppose $\langle w_1, \dots, w_i \rangle \subset \langle v_1, \dots, v_\ell \rangle$. Since v_i are independent, there exist $h \leq \ell$ such that $v_h \notin \langle w_1, \dots, w_i \rangle$. However by (2.9)

$$|q_{i+1}| \leq |(v_h, r)| \leq \mu K^{\tau_1}.$$

Again, by (2.10), this also implies $w_{i+1} \in \langle v_1, \dots, v_\ell \rangle$.

Since the w_i are linearly independent (and so are the v_i), clearly $\langle v_1, \dots, v_\ell \rangle = \langle w_1, \dots, w_\ell \rangle$, so $w_s \in B_K^a \setminus \langle v_1, \dots, v_\ell \rangle$ for $s > j$. This implies that $|q_{j+1}| > \lambda K^{4d\tau_1}$. \blacksquare

Remark 2.5 *With the above lemma we are stating that if m has a (τ_1, ℓ) cut with parameters λ', μ' then, for all choices of $\lambda < \lambda', \mu' < \mu$, there exists a neighborhood B of m such that all points $r \in N$ have a (τ_1, ℓ) cut with parameters λ, μ . The radius of B is determined by the difference in the parameters. In the same way if both $m \xrightarrow{K}[v_i; p_i]$ and $r \xrightarrow{K}[w_i; q_i]$ with $|m + \sigma r| < 5K^3$, have a (τ_1, ℓ) cut with parameters λ, μ then $\langle v_i \rangle_\ell = \langle w_i \rangle_\ell$.*

2.3 Quasi-Töplitz functions

Definition 2.5 *Given K, λ, μ, τ_1 such that $1/2 < \lambda, \mu < 4$, $\tau_0 \leq \tau_1 \leq \tau/4d$ and $4K^3 < \frac{1}{2}K^\tau$ we say that a monomial*

$$e^{i(k, \theta)} I^l z^\alpha \bar{z}^\beta z_m^\sigma z_n^{\sigma'}$$

is $(K, \lambda, \mu, \tau_1)$ -bilinear with the cut $[v_i; p_i]_\ell$ if it satisfies momentum conservation (2.5) i.e.

$$\sigma m + \sigma' n = -\pi(k, \alpha, \beta),$$

$$|k| < K, \quad \min(|n|, |m|) > \lambda K^\tau, \quad \sum_j |j|(\alpha_j + \beta_j) < \mu K^3. \quad (2.11)$$

and moreover there exists $0 \leq \ell < d$ such that both n, m have a (ℓ, τ_1) cut with parameters λ, μ . By convention if $m \xrightarrow{K}[v_i; p_i]$ and $n \xrightarrow{K}[w_i; q_i]$ we suppose that $(p_1, \dots, p_\ell, v_1, \dots, v_\ell) \preceq (q_1, \dots, q_\ell, w_1, \dots, w_\ell)$ and denote the cut by $[v_i; p_i]_\ell$. Notice that by Lemma 2.3 (ii) the cut $[w_i; q_i]_\ell$ is completely fixed by $[v_i; p_i]_\ell$ and $\sigma m + \sigma' n$.

In $\mathcal{A}_{r,s}$ we consider the subspace of $(K, \lambda, \mu, \tau_1)$ -bilinear functions and call $\Pi_{(K, \lambda, \mu, \tau_1)}$ the projection onto this subspace.

Having chosen $1/2, 4$ as bounds for the parameters λ, μ we will call *low momentum variables*, denoted by w^L and spanning the space ℓ_ρ^L , the z_j^σ such that $|j| < 4K^3$. Similarly we call *high momentum variables*, denoted by w^H and spanning the space ℓ_ρ^H , the z_j^σ such that $|j| > K^\tau/2$. Notice that the low and high variables are separated.

For all $\sigma, \sigma' = \pm 1$, $k \in \mathbb{Z}^b$, $h \in Z_1^d$, $\alpha, \beta \in \mathbb{N}^{\mathbb{Z}_1^d}$ and $A \xrightarrow{K}[w_i; q_i]_\ell \in \mathcal{H}_K$ with $|k| < K$, $h = -\pi(k, \alpha, \beta)$, $\sum_{j \in \mathbb{Z}_1^d} |j|(\alpha_j + \beta_j) < \mu K^3$ and $|q_j| < 4K^{\tau/4d}$, let $g_{k, \alpha, \beta}^{\sigma, \sigma'}(h, [w_i; q_i]_\ell; I)$ be an analytic function of I for $|I| < r^2$. We construct Töplitz (K, λ, μ) bilinear functions with a τ_1 cut on the subspace $A \xrightarrow{K}[v_i; p_i]_\ell$

by setting:

$$g(A) := \sum_{n,m,\sigma,\sigma'}^* g^{\sigma,\sigma'}(\sigma m + \sigma' n, [v_i; p_i]_\ell) e^{i\langle k, \theta \rangle} z^\alpha \bar{z}^\beta z_m^\sigma z_n^{\sigma'},$$

where the sum \sum^* means the restriction to the $(K, \lambda, \mu, \tau_1)$ monomials with a (ℓ, τ_1) cut given by $[v_i; p_i]_\ell$ (note that with our convention this means that $m \in A$ moreover each point m is in at most one A). Finally we define $\mathbb{F}(\tau_1, K)$ as the space of functions

$$g = \sum_{\substack{A \in \mathcal{H}_K \\ A \xrightarrow{K} [v_i; p_i]_\ell \mid |p_\ell| < \mu K^{\tau_1}}} g(A).$$

Notice that $\mathbb{F}(\tau_1, K)$ is a subset of the (K, λ, μ) bilinear functions with a τ_1 cut.

Given $f \in \mathcal{A}_{r,s}$ and $\mathcal{F} \in \mathbb{F}(\tau_1, K)$, we define

$$\bar{f}_{\tau_1} := K^{4d\tau_1} (\Pi_{(K,\lambda,\mu,\tau_1)} f - \mathcal{F}), \quad (2.12)$$

Finally set

$$\|X_f\|_{r,s}^T := \sup_{\substack{K \geq \mathcal{K} K \in \mathbb{N}, \\ \tau_0 \leq \tau_1 \leq \tau/4d}} \left[\inf_{\mathcal{F} \in \mathbb{F}(\tau_1, K)} (\max(\|X_f\|_{r,s}, \|X_{\mathcal{F}}\|_{r,s}, \|X_{\bar{f}_{\tau_1}}\|_{r,s})) \right]. \quad (2.13)$$

Definition 2.6 We say that $f \in \mathcal{A}_{r,s}$ is *quasi-Töplitz* of parameters (K, λ, μ) if $\|X_f\|_{r,s}^T < \infty$. We call $\|X_f\|_{r,s}^T$ the *quasi-Töplitz norm* of f .

Remark 2.6 Notice that our definition includes the Töplitz and anti-Töplitz functions, setting $\mathcal{F} = \Pi_{(K,\lambda,\mu,\tau_1)} f$ and hence $\bar{f} = 0$. In the case of Töplitz functions one trivially has $\|X_f\|_{r,s}^T = \|X_f\|_{r,s}$.

3 An abstract KAM theorem

The starting point for our KAM Theorem is a family of Hamiltonians

$$H = N + P, \quad N = \langle \omega(\xi), I \rangle + \sum_{n \in \mathbb{Z}_1^d} \Omega_n(\xi) z_n \bar{z}_n, \quad P = P(\theta, I, z, \bar{z}, \xi). \quad (3.1)$$

defined in $D(r, s) \times \mathcal{O}$, where $\mathcal{O} \subset \mathbb{R}^b$ is open and bounded, say it is contained in a set of diameter D . The functions $\omega(\xi), \Omega_n(\xi)$ are well defined for $\xi \in \mathcal{O}$.

It is well known that, for each $\xi \in \mathcal{O}$, the Hamiltonian equations of motion for the unperturbed N admit the special solutions $(\theta, 0, 0, 0) \rightarrow (\theta + \omega(\xi)t, 0, 0, 0)$ that correspond to invariant tori in the phase space.

Our aim is to prove that, under suitable hypotheses, there is a set $\mathcal{O}_\infty \subset \mathcal{O}$ of positive Lebesgue measure, so that, for all $\xi \in \mathcal{O}_\infty$ the Hamiltonians H still admit invariant tori.

We require the following hypotheses on N, P and \mathcal{O} .

(A1) *Non-degeneracy:* The map $\xi \rightarrow \omega(\xi)$ is a C_W^1 diffeomorphism between \mathcal{O} and its image with $|\omega|_{C_W^1}, |\nabla \omega^{-1}|_{\mathcal{O}} \leq M$.

(A2) *Asymptotics of normal frequency:*

$$\Omega_n(\xi) = |n|^2 + \tilde{\Omega}_n(\xi), \quad (3.2)$$

where $\tilde{\Omega}_n$'s are C_W^1 functions of ξ with C_W^1 -norm uniformly bounded by some positive constant L with $LM < \frac{1}{2}$.

(A3) *Regularity of perturbation:* The perturbation P satisfies momentum conservation, it is real analytic and C_W^1 in $\xi \in \mathcal{O}$. Namely $P \in \mathcal{A}_{r,s}$.

(A4) *Quasi-Töplitz property:* the functions P and $\sum_j \tilde{\Omega}_j |z_j|^2$ are quasi-Töplitz with parameters $(\mathcal{K}, \lambda, \mu)$ where

$$\frac{1}{2} < \lambda, \mu < 4, \quad (\mu - \frac{1}{2})\mathcal{K}^{\tau_0}, (4 - \lambda)\mathcal{K}^{4d\tau_0} > 5\mathcal{K}^4.$$

One has the bounds:

$$\|X_P\|_{D(r,s),\mathcal{O}}^T < \infty, \|\langle \tilde{\Omega}z, z \rangle\|_{D(r,s),\mathcal{O}}^T < L$$

Now we state our infinite dimensional KAM theorem.

Theorem 2 *Assume that Hamiltonian $N + P$ in (3.1) satisfies (A1 – A4). $\gamma > 0$ is small enough, $\varepsilon = \varepsilon(\gamma, b, d, L, M, \mathcal{K}, \lambda, \mu)$ is a positive constant. If $\|X_P\|_{D(r,s),\mathcal{O}}^T \leq \varepsilon$, then there exists a Cantor set $\mathcal{O}_\gamma \subset \mathcal{O}$ with $\text{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma)$ and two maps (analytic in θ and C_W^1 in ξ)*

$$\Psi : \mathbb{T}^b \times \mathcal{O}_\gamma \rightarrow D(r, s), \quad \tilde{\omega} : \mathcal{O}_\gamma \rightarrow \mathbb{R}^b,$$

where Ψ is $\frac{\varepsilon}{\gamma^2}$ -close to the trivial embedding $\Psi_0 : \mathbb{T}^b \times \mathcal{O} \rightarrow \mathbb{T}^b \times \{0, 0, 0\}$ and $\tilde{\omega}$ is ε -close to the unperturbed frequency ω , such that for any $\xi \in \mathcal{O}_\gamma$ and $\theta \in \mathbb{T}^b$, the curve $t \rightarrow \Psi(\theta + \tilde{\omega}(\xi)t, \xi)$ is a linearly stable quasi-periodic solution of the Hamiltonian system governed by $H = N + P$.

4 KAM step

Theorem 2 is proved by an iterative procedure. We produce a sequence of hamiltonians $H_\nu = N_\nu + P_\nu$ and a sequence of symplectic transformations $X_{F_{\nu-1}}^1 H_{\nu-1} := H_\nu$, well defined on a domain $D(r_\nu, s_\nu) \times \mathcal{O}_\nu$. At each step, the perturbation becomes smaller at cost of reducing the analyticity and parameter domain. More precisely, the perturbation should satisfy $\|X_{P_{\nu+1}}\|_{D(r_{\nu+1}, s_{\nu+1}), \mathcal{O}_\nu} \leq \varepsilon_\nu^\kappa, \kappa > 1$. The sequence $r_\nu \rightarrow 0$ while $s_\nu \rightarrow s/4$ and $\mathcal{O}_\nu \rightarrow \mathcal{O}_\infty$. For simplicity of notation, we denote the quantities in the ν -th step without subscript, i.e. $\mathcal{O}_\nu = \mathcal{O}$, $\omega^\nu = \omega$ and so on. The quantities in the $(\nu + 1)^{th}$ step are denoted with subscript “+”. Most of the KAM procedure is completely standard, see [12] for proofs. The new part is: 1. to show that Quasi Töplitz property (A4) for P and $\langle \tilde{\Omega}z, \bar{z} \rangle$ is kept by KAM iteration and 2. prove the measure estimate using the Quasi Töplitz property.

For simple, below we always use C (could be different) denote the constant independent on iteration.

One step Given Hamiltonian (3.1) well defined in $D(r, s) \times \mathcal{O}$ satisfies (A1 – A4). P and $\langle \tilde{\Omega}z, \bar{z} \rangle$ are Quasi Töplitz with parameter $(\mathcal{K}, \lambda, \mu)$. And we have

$$|\omega|_{C_W^1}, |\nabla \omega^{-1}|_{\mathcal{O}} \leq M, |\tilde{\Omega}_n|_{C_W^1} \leq L, \|\langle \tilde{\Omega}z, \bar{z} \rangle\|_{D(r, s), \mathcal{O}}^T \leq L, \|X_P\|_{D(r, s), \mathcal{O}}^T \leq \varepsilon.$$

our aim is to construct: (1) a open set $\mathcal{O}_+ \subset \mathcal{O}$ of positive measure, (2) a 1-parameter group of symplectic transformations Φ_F^t , well defined for all $\xi \in \mathcal{O}_+$, $t \leq 1$, such that $\Phi_F^1 H := H_+ = N_+ + P_+$ still satisfies (A1)–(A4) in domain $D(r_+, s_+)$. P_+ and $\langle \tilde{\Omega}^+ z, \bar{z} \rangle$ are Quasi Töplitz with new parameter $(\mathcal{K}_+, \lambda_+, \mu_+)$. And we have

$$|\omega_+|_{C_W^1}, |\nabla \omega_+^{-1}|_{\mathcal{O}} \leq M_+, |\tilde{\Omega}_n^+|_{C_W^1} \leq L_+, \|\langle \tilde{\Omega}^+ z, \bar{z} \rangle\|_{D(r_+, s_+), \mathcal{O}_+}^T \leq L_+ \\ \|X_{P_+}\|_{D(r_+, s_+), \mathcal{O}_+}^T \leq \varepsilon_+ = \varepsilon^\kappa.$$

Let us define

$$R := \sum_{k, 2|p|+|\alpha|+|\beta| \leq 2} P_{k,p,\alpha,\beta} e^{i(k,\theta)} I^p z^\alpha \bar{z}^\beta, \quad \langle R \rangle := \sum_{i=1}^b P_{0,e_i,0,0} I_i + \sum_{j \in \mathbb{Z}_1^d} P_{0,0,e_j,e_j} |z_j|^2$$

The generating function of our symplectic transformation, denoted by F , solves the “homological equation”:

$$\{N, F\} = \Pi_{\leq \mathcal{K}} R - \langle R \rangle \quad (4.1)$$

where $\Pi_{\leq \mathcal{K}}$ is the projection which collects all terms in R with $|k| \leq \mathcal{K}$ and \mathcal{K} is fixed to be the quasi-Töplitz parameter of $P, \tilde{\Omega}$. It's well known (and immediate) that F is uniquely defined by homological equation for those ξ such that $\langle \omega(\xi), k, \rangle + \Omega(\xi) \cdot l \neq 0$. To have quantitative bounds, we restrict to a set \mathcal{O}_+ where (see Lemma 4.1):

$$|\langle \omega(\xi), k, \rangle + \Omega(\xi) \cdot l| \geq \gamma \mathcal{K}^{-\tau_F}, \quad |k| \leq \mathcal{K}, \quad |l| \leq 2, \quad (k, l) \neq 0, \quad (4.2)$$

where $k \in \mathbb{Z}^b$, $l \in \mathbb{Z}^{\mathbb{Z}_1^d}$ and $(k, l = \alpha - \beta)$ satisfy momentum conservation (2.5); τ_F is a fixed parameter. Then H in the new variables is:

$$H_+ := e^{\{F, \cdot\}} H = N_+ + P_+$$

where $N_+ = N + \langle R \rangle$ and $P_+ = e^{\{F, \cdot\}} H - N_+$.

4.1 The set \mathcal{O}_+

Definition 4.1 \mathcal{O}_+ is defined to be the open subset of \mathcal{O} such that:

i) For all $|k| < \mathcal{K}$, $h \in \mathbb{Z}$, $(h, k) \neq (0, 0)$.

$$|\langle \omega, k \rangle + h| > 2\gamma \mathcal{K}^{-\tau_0}. \quad (4.3)$$

ii) For all $|k| < \mathcal{K}$, $l \in \mathbb{Z}^\infty$, $|l| = 1$.

$$|\langle \omega, k \rangle + \Omega \cdot l| > 2\gamma \mathcal{K}^{-\tau_0}. \quad (4.4)$$

iii) For all $|k| < \mathcal{K}$, $|l| = 2$; $l \neq e_m - e_n$ or $l = e_m - e_n$ and $\max(|m|, |n|) \leq 8\mathcal{K}^\tau$.

$$|\langle \omega, k \rangle + \Omega \cdot l| > 2\gamma \mathcal{K}^{-2d\tau}. \quad (4.5)$$

iv) For all K with $\mathcal{K} \leq K \leq 2\mathcal{K}^{\tau/\tau_0}$, for all affine spaces $[v_i, p_i]_\ell$ in \mathcal{H}_K with $|p_\ell| < K^{\tau/4d}$ we choose a point $m^g \in [v_i; p_i]_\ell^g$. For all such m^g and for all k such that $|k| \leq \mathcal{K}$, we require:

$$|\pm \langle \omega, k \rangle + \Omega_{m^g} - \Omega_{n^g}| > 2\gamma \min(K^{-2d\tau_0}, 2^{-4d}|p_\ell|^{-2d}), \quad (4.6)$$

where $n^g = m^g + \pi(k)$.

By assumption \mathcal{O}_+ is open, and we have

Lemma 4.1 For all $\xi \in \mathcal{O}_+$, for all $k \in \mathbb{Z}^b$, $|k| \leq \mathcal{K}$ and $l \in \mathbb{Z}^{\mathbb{Z}_1^d}$, $|l| \leq 2$ which satisfy momentum conservation, we have

$$|\langle \omega, k \rangle + l \cdot \Omega| \geq \gamma \mathcal{K}^{\tau_F}, \quad \tau_F = 2d\tau. \quad (4.7)$$

Proof: The cases with $|l| = 0, 1$ follow trivially since τ_F is large with respect to τ_0 ; same for $l = e_m + e_n$ and $l = e_m - e_n$ with $\max(|m|, |n|) < 8\mathcal{K}^\tau$.

For the remaining cases we proceed in two steps: first we fix k , $K = \mathcal{K}$ and one subspace $[v_i; p_i]_\ell$, we consider (4.6) with this choice of $k, [v_i; p_i]_\ell$. We show that this inequality implies that (4.7) holds for all $l = e_m - e_n$ such that $m \in [v_i; p_i]_\ell^g$ and $n = m + \pi(k)$. We prove this fact by using the hypothesis that (A4) holds for $\tilde{\Omega}$. Finally we show that every point m with $|m| > 4\mathcal{K}^\tau$ must belong to some $[v_i; p_i]_\ell^g$.

Let m be any point in $[v_i; p_i]_\ell^g$. Let us first notice that

$$\langle \omega, k \rangle + |m|^2 - |n|^2 = \langle \omega, k \rangle + |\pi(k)|^2 - 2\langle \pi(k), m \rangle, \quad (4.8)$$

hence (4.7) with $l = e_m - e_n$ is surely satisfied if $|\langle \pi(k), m \rangle| \geq 2\mathcal{K}^3$ because in that case (4.8) is greater than $2\mathcal{K}^3 - C_1^2\mathcal{K}^2 - |\omega|\mathcal{K} > \mathcal{K}^3$ provided that \mathcal{K} is large with respect to C_1 and $|\omega|$.

If on the other hand $|\langle \pi(k), m \rangle| < 2\mathcal{K}^3$, then $\pi(k) \in B_{\mathcal{K}}^a$ is in $\langle v_i \rangle_\ell$, otherwise we would have $|\langle \pi(k), m \rangle| > \frac{1}{2}\mathcal{K}^{4d\tau_0}$ by definition of $[v_i; p_i]_\ell^g$ and recalling that $\mathcal{K}^{4d\tau_0} > 4\mathcal{K}^3$ by hypothesis. Thus for all $m \in [v_i; p_i]_\ell^g$ either (4.7) is trivially satisfied or

$$|m|^2 - |n|^2 = |\pi(k)|^2 - 2\langle \pi(k), m \rangle = |\pi(k)|^2 - 2\langle \pi(k), m^g \rangle,$$

recall that m^g is one fixed point in $[v_i; p_i]_\ell^g$ on which we have imposed the non-resonance conditions (4.6).

We have seen in Remark 2.5 that all $m \in [v_i; p_i]_\ell^g$ have a (ℓ, τ_1) cut with parameters $(4, \frac{1}{2})$.

We know that $\sum_m \tilde{\Omega}_m |z_m|^2$ is quasi-Töplitz hence, by definition:

$$\begin{aligned} \Pi_{(K, \lambda, \mu, \tau_1)} \sum_m \tilde{\Omega}_m |z_m|^2 = \\ \sum_{\substack{[v_i; p_i]_\ell \in \mathcal{H}(\mathcal{K}) \\ |p_\ell| \leq \mu\mathcal{K}^{\tau_1}}} \sum_m^* \tilde{\Omega}([v_i; p_i]_\ell) |z_m|^2 + K^{-4d\tau_1} \sum_m^* \bar{\Omega}_m |z_m|^2, \end{aligned} \quad (4.9)$$

where \sum_m^* is the restriction to those m with $|m| > \lambda\mathcal{K}^\tau$ and have (ℓ, τ_1) cut with parameters (λ, μ) given by $[v_i; p_i]_\ell$.

All $m \in [v_i; p_i]_\ell^g$ satisfy such conditions with $K = \mathcal{K}$, hence:

$$|\tilde{\Omega}_m - \tilde{\Omega}([v_i; p_i]_\ell)| < L\mathcal{K}^{-4d\tau_1},$$

in particular this relation holds for m^g .

We proceed in the same way for $n = m + \pi(k)$. Let $n \xrightarrow{\mathcal{K}} [w_i; q_i]$. By hypothesis $(\mu - \frac{1}{2})\mathcal{K}_1^\tau, (4 - \lambda)\mathcal{K}^{4d\tau_1} > 5\mathcal{K}^4$ and we apply Lemma 2.3 with $\lambda, \mu, \lambda' = 4, \mu' = \frac{1}{2}$. we produce a (τ_1, ℓ) cut $[w_i; q_i]_\ell$ for n with parameters λ, μ . Now, by Lemma 2.3 ii), $[w_i; q_i]_\ell$ is completely fixed by $[v_i; p_i]_\ell$ and k . We have

$$|\tilde{\Omega}_n - \tilde{\Omega}([w_i; q_i]_\ell)| < L\mathcal{K}^{-4d\tau_1},$$

and this relation holds also for $n^g = m^g + \pi(k)$. This implies that

$$|\tilde{\Omega}_m - \tilde{\Omega}_n - \tilde{\Omega}_{m^g} + \tilde{\Omega}_{n^g}| \leq 4L\mathcal{K}^{-4d\tau_1},$$

where by definition of τ_1 , $\mathcal{K}^{\tau_1} = \max(\mathcal{K}^{\tau_0}, 2|p_\ell|)$ and hence:

$$\begin{aligned} |\langle \omega, k \rangle + \Omega_m - \Omega_n| &\geq |\langle \omega, k \rangle + \Omega_{m^g} - \Omega_{n^g}| - 4L\mathcal{K}^{-4d\tau_1} \geq \\ &\gamma \min(\mathcal{K}^{-4d\tau_0}, 2^{-4d}|p_\ell|^{-4d}) \geq \gamma\mathcal{K}^{-\tau}. \end{aligned} \quad (4.10)$$

We know that each point $m \in \mathbb{Z}_1^d$ admits a “standard cut” based on its \mathcal{K} -optimal presentation $m \xrightarrow{\mathcal{K}} [v_i; p_i]$ (see Definition 2.3). If we have $|p_d| < 4\mathcal{K}^{\frac{\tau}{4d}}$, then by Cramers rule we have $|m| = |V^{-1}p| < 4\mathcal{K}^{\tau/4d}\mathcal{K}^{d-1} < \mathcal{K}^\tau$ and hence $\max(|m|, |m + \pi(k)|) < 2\mathcal{K}^\tau$. So the measure estimates for the points m which fall in this case are covered by (4.5).

If we have $|p_1| > \mathcal{K}^{4d\tau_0}$ then

$$|\pm \langle \omega, k \rangle + \Omega_m - \Omega_n| > |\pm \langle \omega, k \rangle + |m|^2 - |n|^2| - 2L\mathcal{K}^{-4d\tau_0} > \gamma\mathcal{K}^{\tau_0}.$$

Otherwise, there exist $1 \leq j < d$, and $\tau_0 < \tau_1 < \frac{\tau}{4d}$ such that $|p_\ell| < \frac{1}{2}\mathcal{K}^{\tau_1}$ and $|p_{\ell+1}| > 4\mathcal{K}^{4d\tau_1} > 4 \cdot \max(\mathcal{K}^{4d\tau_0}, 2^{4d}|p_\ell|^{4d})$. Thus $m \in [v_i; p_i]_\ell^g$. We have shown that conditions ii)-iv) in \mathcal{O}^+ imply (4.7). \blacksquare

Remark 4.1 *This lemma essentially saying that by improving only one non resonant condition (4.6), we impose **all** the conditions (4.7) with $l = e_m - e_n$ such that $m \in [v_i; p_i]_j^g$ and $n = m + \pi(k)$.*

Remark 4.2 *Notice that up to now we only use (4.6) with $K = \mathcal{K}$. Indeed the other non-resonance conditions are only required in order to show that the quasi-Töplitz property is preserved in solving the homological equation.*

Lemma 4.2 *The set \mathcal{O}_+ is open and has $|\mathcal{O} \setminus \mathcal{O}_+| \leq C\gamma\mathcal{K}^{-\tau_0+b+d/2}$.*

For the measure estimates we define

$$\mathcal{R}_{k,l}^\tau := \left\{ \xi \in \mathcal{O} \mid |\langle \omega, k \rangle + \Omega \cdot l| < \gamma \mathcal{K}^{-\tau} \right\},$$

Lemma 4.3 *For all $(k, l) \neq (0, 0)$ $|k| \leq \mathcal{K}$ and $|l| \leq 2$, which satisfy momentum conservation, one has $|\mathcal{R}_{k,l}^\tau| \leq C\gamma\mathcal{K}^{-\tau}$.*

Proof: By assumption \mathcal{O} is contained in some open set of diameter D . Choose a to be a vector such that $\langle k, a \rangle = |k|$, we have

$$|\partial_t(\langle k, \omega(\xi + ta) \rangle + \Omega \cdot l)| \geq M(|k| - ML) \geq \frac{M}{2}.$$

lead to

$$\int_{\mathcal{R}_{k,l}^\tau} d\xi \leq 2M^{-1}\gamma\mathcal{K}^{-\tau} \int_{\xi+ta \cap \mathcal{R}_{k,l}^\tau} dt \int d\xi_2 \dots d\xi_b \leq 2M^{-1}D^{b-1}\gamma\mathcal{K}^{-\tau}$$

■

Proof of Lemma 4.2:

Proof: The first statement is trivial, indeed $i)-iv)$ are a finite number of inequalities; notice that in $iv)$ for each $[v_i, p_i]_\ell^g$ and k we impose only one condition by choosing one couple m^g, n^g . Finally by Remark 2.2 there are a finite number of $[v_i, p_i]_\ell^g$.

Let us prove the measure estimates; to impose (4.3) with $h = 0$ we have to remove

$$|\cup_{|k| \leq \mathcal{K}} \mathcal{R}_{k,0}^{\tau_0}| \leq C\gamma\mathcal{K}^{-\tau_0+b},$$

the case $h \in \mathbb{Z}$ is exactly the same.

In (4.4), by momentum conservation $l = \pm e_m$ implies that $\pm m = -\pi(k)$. Hence to impose (4.4) we have to remove:

$$|\cup_{k \leq \mathcal{K}} \cup_{\substack{l=\pm e_m, \\ \pm m=-\pi(k)}} \mathcal{R}_{k,l}^{\tau_0}| \leq C\gamma\mathcal{K}^{-\tau_0+b}.$$

If $l = \pm(e_m + e_n)$ we notice that the condition

$$|\pm \langle \omega, k \rangle + |m|^2 + |n|^2 + \tilde{\Omega}_m + \tilde{\Omega}_n| < \frac{1}{2}$$

implies $|\pm \langle \omega, k \rangle + |m|^2 + |n|^2| < 1$ and hence $|m|^2 + |n|^2 < 2|\omega|\mathcal{K}$:

$$|\cup_{k \leq \mathcal{K}} \cup_{\substack{l=\pm(e_m+e_n), \\ m+n=-\pi(k), |m| \leq C(b)\sqrt{\mathcal{K}}}} \mathcal{R}_{k,l}^{\tau_0}| \leq C\gamma\mathcal{K}^{-\tau_0+b+d/2},$$

In conclusion one gets (4.3) and (4.4) with $\tau_0 > b + d/2$ and $l \neq \pm(e_m - e_n)$ by removing an open set of measure $C\gamma\mathcal{K}^{-\tau_0+b+d/2}$.

One trivially has

$$|\cup_{k \leq \mathcal{K}} \cup_{l=\pm(e_m-e_n), m-n=\mp\pi(k), \max(|m|,|n|) \leq 8\mathcal{K}^\tau} \mathcal{R}_{k,l}^{2d\tau}| \leq C\gamma\mathcal{K}^{-d\tau+b},$$

so we have (4.5) by removing an open set of measure $C\gamma\mathcal{K}^{-d\tau+b}$.

To deal with the last case, for all natural K such that $\mathcal{K} \leq K \leq 2\mathcal{K}^{\tau/\tau_0}$, for all affine subspaces $[v_i; p_i]_\ell$ and for all $|k| \leq \mathcal{K}$ we set

$$\mathcal{R}_{k,[v_i;p_i]_\ell}^K := \{\xi \mid |\langle \omega, k \rangle + \Omega_{m^g} - \Omega_{n^g}| < 2\gamma \min(K^{-4d\tau_0}, 2^{-4d}|p_\ell|^{-4d})\} \quad (4.11)$$

Following Lemma 4.3, $|\mathcal{R}_{k,[v_i;p_i]_\ell}^K| < C\gamma \min(K^{-4d\tau_0}, 2^{-4d}|p_\ell|^{-4d})$. By Remark 2.2 we have:

$$\begin{aligned} & |\cup_{\mathcal{K} \leq K \leq \mathcal{K}^{\tau/\tau_0}} \cup_{\ell=0,\dots,d-1} \cup_{\frac{1}{2}K^{\tau_0} \leq |p_\ell| \leq 4K^{\frac{\tau}{4d}}} \cup_{\substack{[v_i;p_i]_\ell^g \\ |k| < \mathcal{K}}} \mathcal{R}_{k,[v_i;p_i]_\ell}^K| \\ & \leq C\gamma \sum_{K \geq \mathcal{K}} \sum_{\ell=0}^{d-1} \sum_{|p_\ell| > \frac{1}{2}K^{\tau_0}} |p_\ell|^{-4d-1+d} K^{\ell d} \mathcal{K}^b \leq 4^d C_2 \gamma \mathcal{K}^{-d\tau_0+b}, \end{aligned}$$

so that we have (4.6) by removing an open set of measure $C\gamma\mathcal{K}^{-d\tau_0+b}$.

Our Lemma is proved automatically. \blacksquare

4.2 Quasi-Toplitz property

Proposition 1 *The functions $P_+, \tilde{\Omega}^+|z|^2$ are quasi-Töplitz with parameters $(\mathcal{K}_+, \lambda_+, \mu_+)$ such that:*

$$4\mathcal{K}_+ < \sqrt{(\mu - \mu_+)(\mathcal{K}_+)^{3/2}}, \quad 4\mu_+\mathcal{K}_+^4 < (\lambda_+ - \lambda)\mathcal{K}_+^{4d\tau_0-1}.$$

The key of our strategy is based on the following two propositions which are proved in the appendix.

Proposition 2 *For all $K \geq \mathcal{K}$, $k \in \mathbb{Z}^b$ with $|k| < \mathcal{K}$ and for all $|m|, |n| \geq \lambda K^\tau$ such that $m-n = -\pi(k)$, $m \xrightarrow{K} [v_i; p_i]$, $n \xrightarrow{K} [w_i; q_i]$ and m, n have a (ℓ, τ_1) cut with parameters λ, μ for some choice of ℓ, τ_1 one has*

$$\begin{aligned} & |\langle \omega, k \rangle + |m|^2 - |n|^2 + \tilde{\Omega}([v_i; p_i]_\ell) - \tilde{\Omega}([w_i; q_i]_\ell)| = \\ & |\langle \omega, k \rangle + |\pi(k)|^2 - 2\langle \pi(k), m \rangle + \tilde{\Omega}([v_i; p_i]_\ell) - \tilde{\Omega}([w_i; q_i]_\ell)| \geq \\ & \begin{cases} \gamma\mathcal{K}^{-2d\tau\tau_1/\tau_0}, & \pi(k) \in \langle v_i \rangle_\ell \\ \frac{1}{2}K^{4d\tau_1}, & \text{otherwise} \end{cases}, \end{aligned}$$

where $\tilde{\Omega}([v_i; p_i]_\ell)$ and $\tilde{\Omega}([w_i; q_i]_\ell)$ are defined by Formula (4.9).

Proposition 3 For $\xi \in \mathcal{O}_+$, the solution of the homological equation F is quasi-Töplitz for parameters $(\mathcal{K}, \lambda, \mu)$, moreover one has the bound

$$\|X_F\|_{r,s}^T \leq C\gamma^{-2}\mathcal{K}^{2\tau^2/\tau_0}\|X_P\|_{r,s}^T, \quad (4.12)$$

where C is some constant.

Analytic quasi-Töplitz functions are closed under Poisson bracket. More precisely:

Proposition 4 Given $f^{(1)}, f^{(2)} \in \mathcal{A}_{r,s}$, quasi-Töplitz with parameters $(\mathcal{K}, \lambda, \mu)$ we have that $\{f^{(1)}, f^{(2)}\} \in \mathcal{A}_{r',s'}$, is quasi-Töplitz for all parameters $(\mathcal{K}', \lambda', \mu')$ such that $\mathcal{K}', \lambda', \mu', r', s'$ satisfy:

$$\begin{aligned} \mathcal{K}' &< (\mu - \mu')\mathcal{K}'^3, \quad 2\mu'\mathcal{K}'^3 < (\lambda' - \lambda)\mathcal{K}'^{4d\tau_0-1}, \\ e^{-(s-s')\mathcal{K}'}\mathcal{K}'^\tau &< 1. \end{aligned} \quad (4.13)$$

We have the bounds

$$\|X_{\{f^{(1)}, f^{(2)}\}}\|_{r',s'}^T \leq C\delta^{-1}\|X_{f^{(1)}}\|_{r,s}^T\|X_{f^{(2)}}\|_{r,s}^T \quad (4.14)$$

where $\delta = \min(s - s', 1 - \frac{r'}{r})$

(ii) Given $f^{(1)}, f^{(2)}$ as in item (i), with $\|X_{f^{(1)}}\|_{r,s}^T\delta^{-1} \ll 1$, the function $f^{(2)} \circ \phi_{f^{(1)}}^t := e^{t\{f^{(1)}, \cdot\}}f^{(2)}$, for $t \leq 1$, is quasi-Töplitz in $\mathcal{D}(r', s')$ with parameters $(\mathcal{K}', \lambda', \mu')$ for

$$\mathcal{K}' \leq \sqrt{(\mu - \mu')}\mathcal{K}'^{3/2}, \quad 2\mu'\mathcal{K}'^4 < (\lambda' - \lambda)\mathcal{K}'^{4d\tau_0-1}. \quad (4.15)$$

5 Estimate and KAM Iteration

5.1 Estimate on the coordinate transformation

We estimate X_F and ϕ_F^1 where F is given by (4.1).

Lemma 5.1 Let $D_i = D(\frac{i}{4}r, s_+ + \frac{i}{4}(s - s_+))$, $0 < i \leq 4$. Then

$$\|X_F\|_{D_3, \mathcal{O}_+} \leq c\gamma^{-2}\mathcal{K}^{4d\tau}\varepsilon, \quad \|X_F\|_{D_3, \mathcal{O}_+}^T \leq C\gamma^{-2}\mathcal{K}^{2\tau^2/\tau_0}\varepsilon \quad (5.1)$$

Lemma 5.2 *Let $\eta = \varepsilon^{\frac{1}{3}}$, $D_{i\eta} = D(\frac{i}{4}\eta r, s_+ + \frac{i}{4}(s - s_+))$, $0 < i \leq 4$. If $\varepsilon \ll (\frac{1}{2}\gamma^2\mathcal{K}^{-2\tau^2/\tau_0})^{\frac{3}{2}}$, we then have*

$$\phi_F^t : D_{2\eta} \rightarrow D_{3\eta}, \quad -1 \leq t \leq 1, \quad (5.2)$$

Moreover,

$$\|D\phi_F^t - Id\|_{D_{1\eta}} \leq C\gamma^{-2}\mathcal{K}^{4d\tau}\varepsilon, \quad \|D\phi_F^t - Id\|_{D_{1\eta}}^T \leq C\gamma^{-2}\mathcal{K}^{2\tau^2/\tau_0}\varepsilon \quad (5.3)$$

The first bound in the two lemmas is completely standard and corresponds respectively to Lemma 4.2 and 4.3 in [12], we refer to that paper for the proof. Let us fix \mathcal{K}_+ so that (4.15) holds. Let us prove the quasi-Töplitz bound in Lemma 5.1. This follows by Formula (4.12) of Proposition 1 with $\mathcal{K}' = \mathcal{K} + \frac{3}{4}(\mathcal{K}_+ - \mathcal{K})$, $\lambda' = \lambda + \frac{3}{4}(\lambda_+ - \lambda)$ and $\mu' = \mu - \frac{3}{4}(\mu_+ - \mu)$:

5.2 Estimate of the new perturbation

The symplectic map ϕ_F^1 defined above transforms H into $H_+ = N_+ + P_+$, where $N_+ = N + \langle R \rangle$ and

$$\begin{aligned} P_+ &= \int_0^1 (1-t) \{ \{N, F\}, F \} \circ \phi_F^t dt + \int_0^1 \{ \Pi_{\leq \mathcal{K}} R, F \} \circ \phi_F^t dt + (P - \Pi_{\leq \mathcal{K}} R) \circ \phi_F^1 \\ &= \int_0^1 \{ R(t), F \} \circ \phi_F^t dt + (P - \Pi_{\leq \mathcal{K}} R) \circ \phi_F^1, \end{aligned} \quad (5.4)$$

with $R(t) = (1-t)(N_+ - N) + t\Pi_{\leq \mathcal{K}} R$. Hence

$$X_{P_+} = \int_0^1 (\phi_F^t)^* X_{\{R(t), F\}} dt + (\phi_F^1)^* X_{(P - \Pi_{\leq \mathcal{K}} R)}.$$

Lemma 5.3 *The new perturbation P_+ satisfies the estimate*

$$\|X_{P_+}\|_{D(r_+, s_+)} \leq C\gamma^{-2}\mathcal{K}^{4d\tau}\varepsilon^{4/3}.$$

Proof: According to Lemma 5.2,

$$\|D\phi_F^t - Id\|_{D_{1\eta}} \leq c\gamma^{-2}\mathcal{K}^{4d\tau}\varepsilon, \quad -1 \leq t \leq 1,$$

thus

$$\|D\phi_F^t\|_{D_{1\eta}} \leq 1 + \|D\phi_F^t - Id\|_{D_{1\eta}} \leq 2, \quad -1 \leq t \leq 1.$$

$$\|X_{\{R(t), F\}}\|_{D_{2\eta}} \leq C\gamma^{-2}\mathcal{K}^{4d\tau}\eta^{-2}\varepsilon^2,$$

and

$$\|X_{(P-\Pi_{\leq \mathcal{K}}R)}\|_{D_{2\eta}} \leq C\eta\varepsilon,$$

we have

$$\|X_{P_+}\|_{D(r_+, s_+)} \leq C\eta\varepsilon + C(\gamma^{-2}\mathcal{K}^{4d\tau})\eta^{-2}\varepsilon^2 \leq C\gamma^{-2}\mathcal{K}^{4d\tau}\varepsilon^{4/3}.$$

■

We need to show that P_+ is quasi-Töplitz and estimate its Töplitz norm. We notice that $R(t)$ and $P - \Pi_{\leq \mathcal{K}}R$ in (5.4) are quasi-Töplitz, by hypothesis (A4). Then, by Proposition 4 *ii*), we have that $R(t) \circ \phi_F^t = e^{\{F, \cdot\}}R(t)$ and $(P - \Pi_{\leq \mathcal{K}}R) \circ \phi_F^t$ are quasi-Töplitz as well.

We repeat the reasoning of Lemma 5.3 only with the Töplitz norm. We have the following

Lemma 5.4 *We set $\varepsilon_+ := C\gamma^{-2}\mathcal{K}^{2\tau^2/\tau_0}\varepsilon^{4/3}$ for an appropriate constant C , we have*

$$\|X_{P_+}\|_{D(r_+, s_+)}^T \leq \varepsilon_+.$$

5.3 Iteration lemma

To make KAM machine work fluently, a sequence of iteration is given: For any given $s, \varepsilon, r, \gamma$ and for all $\nu \geq 1$, we define the following sequences

$$\begin{aligned} s_\nu &= s(1 - \sum_{i=2}^{\nu+1} 2^{-i}), \\ \varepsilon_\nu &= c\gamma^{-2}\mathcal{K}_{\nu-1}^{2\tau^2/\tau_0}\varepsilon_{\nu-1}^{\frac{4}{3}}, \\ \gamma_\nu &= \gamma(1 - \sum_{i=2}^{\nu+1} 2^{-i}), \quad \eta_\nu = \varepsilon_\nu^{\frac{1}{3}} \\ M_\nu &= M_{\nu-1} + \varepsilon_{\nu-1}, \quad L_\nu = L_{\nu-1} + \varepsilon_{\nu-1}, \\ r_\nu &= \frac{1}{4}\eta_{\nu-1}r_{\nu-1} = 2^{-2\nu}(\prod_{i=0}^{\nu-1} \varepsilon_i)^{\frac{1}{3}}r_0, \\ \mu_\nu &= \mu - \sum_{i=1}^{\nu} (\chi)^{-i}, \quad \lambda_\nu = \lambda + \sum_{i=1}^{\nu} (\chi)^{-i} \\ \rho_\nu &= \rho(1 - \sum_{i=2}^{\nu+1} 2^{-i}), \end{aligned} \tag{5.5}$$

$$\mathcal{K}_\nu = c(\rho_{\nu-1} - \rho_\nu)^{-1} \ln \varepsilon_\nu^{-1},$$

where $c, 1 < \chi < \frac{4}{3}$ is a constant, and the parameters $r_0, \varepsilon_0, \gamma_0, L_0, s_0$ and \mathcal{K}_0 are defined to be $r, \varepsilon, \gamma, L, s$ and bounded by $\ln \varepsilon^{-1}$ respectively.

We iterate the KAM step, and proceed by induction.

Lemma 5.5 *Suppose at the ν -step of KAM iteration, hamiltonian*

$$H_\nu = N_\nu + P_\nu,$$

is well defined in $D(r_\nu, s_\nu) \times \mathcal{O}_\nu$, where N_ν is usual "integrable normal form", P_ν and $\sum \tilde{\Omega}_n^\nu |z_n|^2$ satisfy (A4) for $(\mathcal{K}_\nu, \lambda_\nu, \mu_\nu)$, ω_ν and Ω_n^ν are C_W^1 smooth

$$|\omega_\nu|_{C_W^1}, |\nabla \omega_\nu^{-1}|_{\mathcal{O}} \leq M_\nu, |\tilde{\Omega}_n^\nu|_{C_W^1} \leq L_\nu, \quad |\Omega_n^\nu - \Omega_n^{\nu-1}|_{\mathcal{O}_\nu} \leq \varepsilon_{\nu-1};$$

$$\|X_{P_\nu}\|_{D(r_\nu, s_\nu), \mathcal{O}_\nu}^T \leq \varepsilon_\nu, \quad \|\langle \tilde{\Omega}^\nu z, \bar{z} \rangle\|_{D(r_\nu, s_\nu), \mathcal{O}_\nu}^T \leq L_\nu$$

Then there exists a symplectic and Quasi-Töplitz change of variables for parameter $(\mathcal{K}_{\nu+1}, \lambda_{\nu+1}, \mu_{\nu+1})$,

$$\Phi_\nu : D(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_{\nu+1} \rightarrow D(r_\nu, s_\nu), \quad (5.6)$$

where $|\mathcal{O}_{\nu+1} \setminus \mathcal{O}_\nu| \leq \gamma \mathcal{K}^{-\tau_0 + b + \frac{d}{2}}$, such that on $D(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_{\nu+1}$ we have

$$H_{\nu+1} = H_\nu \circ \Phi_\nu = e_{\nu+1} + N_{\nu+1} + P_{\nu+1} = e_{\nu+1} + \langle \omega_{\nu+1}, I \rangle + \langle \Omega^{\nu+1} z, \bar{z} \rangle + P_{\nu+1},$$

with $\omega_{\nu+1} = \omega_\nu + \sum_{|l|=1} l P_{0,l,0,0}$, $\Omega_n^{\nu+1} = \Omega_n^\nu + P_{0,0,e_n,e_n}^\nu$.

$N_{\nu+1}$ is "integrable normal form". $P_{\nu+1}$ and $\sum \tilde{\Omega}_n^{\nu+1} |z_n|^2$ satisfy (A4) for parameters $(\mathcal{K}_{\nu+1}, \lambda_{\nu+1}, \mu_{\nu+1})$. Functions $\omega_{\nu+1}$ and $\Omega_n^{\nu+1}$ are C_W^1 smooth

$$|\omega_{\nu+1}|_{C_W^1}, |\nabla \omega_{\nu+1}^{-1}|_{\mathcal{O}} \leq M_{\nu+1}, |\tilde{\Omega}_n^{\nu+1}|_{C_W^1} \leq L_{\nu+1}, \quad |\Omega_n^{\nu+1} - \Omega_n^\nu|_{\mathcal{O}_{\nu+1}} \leq \varepsilon_\nu;$$

$$\|X_{P_{\nu+1}}\|_{D(r_{\nu+1}, s_{\nu+1}), \mathcal{O}_{\nu+1}}^T \leq \varepsilon_{\nu+1}, \quad \|\langle \tilde{\Omega}^{\nu+1} z, \bar{z} \rangle\|_{D(r_{\nu+1}, s_{\nu+1}), \mathcal{O}_{\nu+1}}^T \leq L_{\nu+1}$$

• By Proposition 1, new perturbation $P_{\nu+1}$ and $\langle \tilde{\Omega}^{\nu+1} z, z \rangle$ satisfies Quasi-Töplitz property for parameters $(\mathcal{K}_{\nu+1}, \lambda_{\nu+1}, \mu_{\nu+1})$. As we can see, when we require $\tau > \tau_0 > 12$,

$$\forall K \geq \mathcal{K}_{\nu+1} = c(\rho_{\nu-1} - \rho_\nu)^{-1} \ln \varepsilon_\nu^{-1} > \mathcal{K}_0 2^\nu$$

implies inequality

$$2K \leq \sqrt{(\mu_\nu - \mu_{\nu+1})} K^{3/2}, \quad 4\mu' K^4 < (\lambda_{\nu+1} - \lambda_\nu) K^{4d\tau_0 - 1}.$$

• Poisson bracket preserve momentum conservation or result from Lemma 4.4 in [12] lead to $P_{\nu+1}$ satisfies momentum conservation.

5.4 Convergence

Suppose that the assumptions of Theorem 2 are satisfied. Recall

$$\varepsilon_0 = \varepsilon, r_0 = r, \gamma_0 = \gamma, s_0 = s, M_0 = M, L_0 = L, N_0 = N, P_0 = P,$$

\mathcal{O} is a compact set. The assumptions of the iteration lemma are satisfied when $\nu = 0$ if ε_0, γ_0 are sufficiently small. Inductively, we obtain sequences:

$$\mathcal{O}_{\nu+1} \subset \mathcal{O}_\nu,$$

$$\Psi^\nu = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_\nu : D(r_{\nu+1}, s_{\nu+1}) \times \mathcal{O}_{\nu+1} \rightarrow D(r_0, s_0), \nu \geq 0,$$

$$H \circ \Psi^\nu = H_{\nu+1} = N_{\nu+1} + P_{\nu+1}.$$

Let $\tilde{\mathcal{O}} = \bigcap_{\nu=0}^{\infty} \mathcal{O}_\nu$, since at ν step the parameter we excluded is bounded by $C\gamma\mathcal{K}_\nu^{-\tau_0+b+d/2}$, the total measure we excluded with infinity step of KAM iteration is bounded by γ which guarantee $\tilde{\mathcal{O}}$ is a nonempty set, actually it has positive measure.

As in [19, 20], with Lemma 5.2, $N_\nu, \Psi^\nu, D\Psi^\nu, \omega_\nu$ converge uniformly on $D(0, \frac{s}{2}) \times \tilde{\mathcal{O}}$ with

$$N_\infty = e_\infty + \langle \omega_\infty, I \rangle + \sum_n \Omega_n^\infty z_n \bar{z}_n.$$

Since $\mathcal{K}_\nu = c(\rho_{\nu-1} - \rho_\nu)^{-1} \ln \varepsilon_\nu^{-1}$, we have $\varepsilon_\nu = c\gamma^2 \mathcal{K}_{\nu-1}^{\frac{2\tau^2}{\tau_0}} \varepsilon_{\nu-1}^{\frac{4}{3}} \rightarrow 0$ once ε is sufficiently small. And with this we have ω_∞ is slightly different from ω .

Let ϕ_H^t be the flow of X_H . Since $H \circ \Psi^\nu = H_{\nu+1}$, there is

$$\phi_H^t \circ \Psi^\nu = \Psi^\nu \circ \phi_{H_{\nu+1}}^t. \quad (5.7)$$

The uniform convergence of $\Psi^\nu, D\Psi^\nu, \omega_\nu$ and X_{H_ν} implies that the limits can be taken on both sides of (5.7). Hence, on $D(0, \frac{s}{2}) \times \tilde{\mathcal{O}}$ we get

$$\phi_H^t \circ \Psi^\infty = \Psi^\infty \circ \phi_{H_\infty}^t \quad (5.8)$$

and

$$\Psi^\infty : D(0, \frac{s}{2}) \times \tilde{\mathcal{O}} \rightarrow D(r, s) \times \mathcal{O}.$$

From 5.8, for $\xi \in \tilde{\mathcal{O}}$, $\Psi^\infty(\mathbb{T}^b \times \{\xi\})$ is an embedded torus which is invariant for the original perturbed Hamiltonian system at $\xi \in \tilde{\mathcal{O}}$. The normal behavior of this invariant tori is governed by normal frequency Ω_∞ . \blacksquare

A Appendix

Proof of Proposition 2:

By hypothesis

$$\begin{aligned} \min(|m|, |n|) &\geq \lambda K^\tau, \quad m \xrightarrow{K} [v_i; p_i], \quad n \xrightarrow{K} [w_i; q_i], \\ |q_\ell|, |p_\ell| &\leq \mu K^{\tau_1}, \quad |q_{\ell+1}|, |p_{\ell+1}| \geq \lambda K^{4d\tau_1}, \quad [v_i; p_i]_\ell \prec [w_i; q_i]_\ell \end{aligned} \quad (\text{A.1})$$

By definition of quasi-Töplitz (see formula (4.9)), one has:

$$|\tilde{\Omega}_m - \tilde{\Omega}([v_i; p_i]_\ell)|, |\tilde{\Omega}_n - \tilde{\Omega}([w_i; q_i]_\ell)| \leq LK^{-4d\tau_1} \quad (\text{A.2})$$

Recall that $m - n = -\pi(k)$, so one has

$$|m|^2 - |n|^2 = \langle m + n, m - n \rangle = |\pi(k)|^2 - 2\langle \pi(k), m \rangle.$$

If $\pi(k) \notin \langle v_i \rangle_\ell$ then $|\langle \pi(k), m \rangle| > K^{4d\tau_1} > \mathcal{K}^3$ and the *denominator* is not small:

$$|\langle \omega, k \rangle + |m|^2 - |n|^2 + \tilde{\Omega}([v_i; p_i]_\ell) - \tilde{\Omega}([w_i; q_i]_\ell)| > \frac{1}{2} K^{4d\tau_1},$$

since (again by definition of quasi-Töplitz) $|\tilde{\Omega}([v_i; p_i]_\ell)|, |\tilde{\Omega}([w_i; q_i]_\ell)| \leq L$.

If $\pi(k) \in \langle v_i \rangle_\ell$ then the value of $\langle \pi(k), m \rangle$ is fixed for all $m \in [v_i; p_i]_\ell$.

We know that $m \xrightarrow{\mathcal{K}} [v'_i; p'_i]$ has a standard cut, so that $m \in [v'_i; p'_i]_{\bar{\ell}}^g$ for some $\bar{\ell}$. If $2^{4d}\mathcal{K}^\tau < K^{\tau_0}$ then

$$\begin{aligned} &|\langle \omega, k \rangle + |\pi(k)|^2 - 2\langle \pi(k), m \rangle + \tilde{\Omega}([v_i; p_i]_\ell) - \tilde{\Omega}([w_i; q_i]_\ell)| \\ (\text{A.2}) \quad &\geq |\langle \omega, k \rangle + \Omega_m - \Omega_n| - 2LK^{-4d\tau_1} \\ (4.10) \quad &\geq \gamma \min(\mathcal{K}^{-4d\tau_0}, 2^{-4d}|p'_\ell|^{-4d}) - 2L|K|^{-4d\tau_1} \geq \frac{\gamma}{2} \min(\mathcal{K}^{-4d\tau_0}, |p'_\ell|^{-4d}), \end{aligned}$$

since $|p'_\ell| < 4\mathcal{K}^{\tau/4d}$ by the definition of standard cut.

If on the other hand we have $2^{4d}\mathcal{K}^\tau > K^{\tau_0}$ we proceed as follows. We have seen that we may restrict to the case $\pi(k) \in \langle v_i \rangle_j$, where

$$|m|^2 - |n|^2 = |\pi(k)|^2 - 2\langle \pi(k), m \rangle = |\pi(k)|^2 - 2\langle \pi(k), m^g \rangle,$$

where (notice that $K < 2\mathcal{K}^{\tau/\tau_0}$), $m^g := m^g(K)$ is the point in $[v_i; p_i]_\ell^g$ chosen for the measure estimates (4.6).

We notice that m^g, n^g satisfy the conditions (A.1), so we apply (A.2) to m, n, m^g, n^g . We have

$$\begin{aligned} & |\langle \omega, k \rangle + |\pi(k)|^2 - 2\langle \pi(k), m \rangle + \tilde{\Omega}([v_i; p_i]_\ell) - \tilde{\Omega}([w_i; q_i]_\ell)| \geq \\ & |\langle \omega, k \rangle + \Omega_{m^g} - \Omega_{n^g}| - 2LK^{-4d\tau_1} \\ & \geq 2\gamma \min(K^{-4d\tau_0}, 2^{-2d}|p_\ell|^{-4d}) - 2LK^{-4d\tau_1} \geq \\ & \gamma \min(K^{-4d\tau_0}, 2^{-2d}|p_\ell|^{-4d}) \geq \gamma \mathcal{K}^{\frac{-4d\tau_1\tau}{\tau_0}} \end{aligned}$$

since by definition $|p_j| < \mu' K^{\tau_1} < 4K^{\tau_1}$, $K \leq 2\mathcal{K}^{\tau/\tau_0}$ and $2 \cdot 8^{4d}L < \gamma$. \blacksquare

Proof of Proposition 3: The quasi-Töplitz property is a condition on the (K, λ, μ) -bilinear part of F , where F is at most quadratic. Hence we only need to consider the quadratic terms:

$$\Pi_{(K, \lambda, \mu)} F = \sum_{\substack{|k| \leq \mathcal{K}, \\ \max(|n|, |m|) > \lambda' K^\tau}} e^{i(k, x)} (F_{k, 0, e_m, e_n} z_m \bar{z}_n + F_{k, 0, e_m + e_n, 0} z_m z_n) + \text{c.c}$$

with

$$F_{k, 0, e_m, e_n} = \frac{P_{k, 0, e_m, e_n}}{\langle k, \omega \rangle + \Omega_m - \Omega_n}, \quad F_{k, 0, e_m + e_n, 0} = \frac{P_{k, 0, e_m + e_n, 0}}{\langle \omega, k \rangle + \Omega_m + \Omega_n}. \quad (\text{A.3})$$

By hypothesis $\min(|m|, |n|) > \lambda K^\tau$ so in the case of $F_{k, 0, e_m + e_n, 0}$ one has

$$|F_{k, 0, e_m + e_n, 0}| = \frac{|P_{k, 0, e_m + e_n, 0}|}{\langle k, \omega \rangle + |m|^2 + |n|^2 + \tilde{\Omega}_m + \tilde{\Omega}_n} \leq |P_{k, 0, e_m + e_n, 0}| K^{-\tau},$$

since

$$\langle k, \omega \rangle + |m|^2 + |n|^2 + \tilde{\Omega}_m + \tilde{\Omega}_n > 2K^\tau - c\mathcal{K} - 2L.$$

We proceed in the same way for $\partial_\xi F_{k, 0, e_m + e_n, 0}$. This means that $F_{k, 0, e_m + e_n, 0}$ is quasi-Töplitz with the ‘‘Töplitz approximation’’ equal to zero.

Let $m \xrightarrow{K} [v_i; p_i]$ have a cut (ℓ, τ_1) . We wish to show that

$$F_{k, 0, e_m, e_n} = \mathcal{F}_k(m - n, [v_i; p_i]_\ell) + K^{-4d\tau_1} \bar{F}_{k, 0, e_m, e_n},$$

here \mathcal{F}_k is the k Fourier coefficient of the Töplitz approximation \mathcal{F} . By hypothesis we have conditions (A.1) and $\langle v_1, \dots, v_\ell \rangle = \langle w_1, \dots, w_\ell \rangle$. This in turn implies that the subspace $[w_i, q_i]_\ell$ is obtained from $[v_i, p_i]_\ell$ by translation by $m - n = -\pi(k)$. If $\pi(k) \notin \langle v_i \rangle_\ell$ then the denominator in the first of (A.3) is

$$|\langle k, \omega \rangle + \Omega_m - \Omega_n| > |\langle k, \omega \rangle + |m|^2 - |n|^2 + \tilde{\Omega}([v_i; p_i]_\ell) - \tilde{\Omega}([w_i; q_i]_\ell)| - L > \frac{1}{4} K^{4d\tau_1}$$

and we may again set $\mathcal{F}_k = 0$. Otherwise we set

$$\mathcal{F}_k(m-n, [v_i, p_i]_\ell) = \frac{\mathcal{P}_k(m-n, [v_i, p_i]_\ell)}{\langle \omega, k \rangle + |\pi(k)|^2 - 2\langle \pi(k), m \rangle + \tilde{\Omega}([v_i; p_i]_\ell) - \tilde{\Omega}([w_i; q_i]_\ell)}.$$

We notice that $\langle \pi(k), m \rangle$ depends only on the subspace $[v_i, p_i]_\ell$ and on $\pi(k)$. Finally we apply Proposition 2 to bound the denominator. To bound the derivatives in ξ of F we proceed in the same way, only the denominators may appear to the power two. In conclusion:

$$\|X_F\|_{r,s}^T \leq C\mathcal{K}^{\frac{2\pi^2}{\tau_0}} \|X_P\|_{r,s}^T$$

■

Before we give a proof to proposition 4, for simple and better to understand, some notation and technical Lemma are given.

Let us set up some notation. We divide the Poisson bracket in four terms: $\{\cdot, \cdot\} = \{\cdot, \cdot\}^{I,\theta} + \{\cdot, \cdot\}^L + \{\cdot, \cdot\}^H + \{\cdot, \cdot\}^R$ where the apices identify the variables in which we are performing the derivatives (the apex R summarizes the derivatives in all the w_i which are neither low nor high momentum). We call a monomial

$$e^{i(k,\theta)} I^l z^\alpha \bar{z}^\beta$$

1. of (K, μ) -low momentum if $|k| < K$ and $\sum_j |j|(\alpha_j + \beta_j) < \mu K^3$. Denote by $\Pi_{K,\mu}^L$ the projection on this subspace.

2. of K -high frequency if $|k| \geq K$. Denote Π_K^U the projection on this subspace.

Recall projection symbol $\Pi_{K,\lambda,\mu,\tau_1}$ is given in definition 2.5. A function f then may be uniquely represented as $f = \Pi_{K,\lambda,\mu,\tau_1} f + \Pi_{K,\mu}^L f + \Pi_K^U f + \Pi_R f$ where $\Pi_R f$ is by definition the projection on those monomials which are neither $(K, \lambda, \mu, \tau_1)$ bilinear nor of (K, μ) -low momentum nor of K -high frequency.

A technical lemma is given below.

Lemma A.1 *The following splitting formula holds:*

$$\begin{aligned} \Pi_{K,\lambda',\mu',\tau_1} \{f^{(1)}, f^{(2)}\} &= \Pi_{K,\lambda',\mu',\tau_1} \left(\{\Pi_{K,\lambda,\mu,\tau_1} f^{(1)}, \Pi_{K,\lambda,\mu,\tau_1} f^{(2)}\}^H + \right. \\ &\quad \left. \{\Pi_{K,\lambda,\mu,\tau_1} f^{(1)}, \Pi_{K,2\mu}^L f^{(2)}\}^{I,\theta} + \{\Pi_{K,\lambda,\mu,\tau_1} f^{(1)}, \Pi_{K,2\mu}^L f^{(2)}\}^L + \{\Pi_K^U f^{(1)}, f^{(2)}\} \right. \\ &\quad \left. \{\Pi_{K,2\mu}^L f^{(1)}, \Pi_{K,\lambda,\mu,\tau_1} f^{(2)}\}^{I,\theta} + \{\Pi_{K,2\mu}^L f^{(1)}, \Pi_{K,\lambda,\mu,\tau_1} f^{(2)}\}^L + \{f^{(1)}, \Pi_K^U f^{(2)}\} \right) \end{aligned} \quad (\text{A.4})$$

Proof: We perform a case analysis: we replace each $f^{(i)}$ with a single monomial to show which terms may contribute non trivially to the projection $\Pi_{K,\lambda',\mu',\tau_1}\{f^{(1)}, f^{(2)}\}$.

Consider the expression

$$\Pi_{K,\lambda',\mu',\tau_1}\{e^{i(k^{(1)},\theta)}I^{l^{(1)}}z^{\alpha^{(1)}}\bar{z}^{\beta^{(1)}}, e^{i(k^{(2)},\theta)}I^{l^{(2)}}z^{\alpha^{(2)}}\bar{z}^{\beta^{(2)}}\}.$$

If one or both of the $|k^{(i)}| > K$ then one or both monomials are of high frequency and we obtain the last term in the second and third line of (A.4).

Suppose now that $|k^{(1)}|, |k^{(2)}| < K$ we wish to understand under which conditions on the $\alpha^{(i)}, \beta^{(i)}$ this expression is not zero. By direct inspection, one of the following situations (apart from a trivial permutation of the indexes 1, 2) must hold:

1. one has $z^{\alpha^{(1)}}\bar{z}^{\beta^{(1)}} = z^{\bar{\alpha}^{(1)}}\bar{z}^{\bar{\beta}^{(1)}}z_m^\sigma z_j^{\sigma_1}$ and $z^{\alpha^{(2)}}\bar{z}^{\beta^{(2)}} = z^{\bar{\alpha}^{(2)}}\bar{z}^{\bar{\beta}^{(2)}}z_n^{\sigma'} z_j^{-\sigma_1}$, where $\min(|m|, |n|) \geq \lambda'K^\tau$ have a (ℓ, τ_1) cut for some ℓ and $z^{\bar{\alpha}^{(1)}}\bar{z}^{\bar{\beta}^{(1)}}z^{\bar{\alpha}^{(2)}}\bar{z}^{\bar{\beta}^{(2)}}$ is of (K, μ') -low momentum. The derivative in the Poisson bracket is on w_j .
2. one has $z^{\alpha^{(1)}}\bar{z}^{\beta^{(1)}} = z^{\bar{\alpha}^{(1)}}\bar{z}^{\bar{\beta}^{(1)}}z_m^\sigma z_n^{\sigma'}$ and $z^{\alpha^{(2)}}\bar{z}^{\beta^{(2)}} = z^{\bar{\alpha}^{(2)}}\bar{z}^{\bar{\beta}^{(2)}}z_n^{\sigma'}$, where $\min(|m|, |n|) \geq \lambda'K^\tau$ have a (ℓ, τ_1) cut for some ℓ and $z^{\bar{\alpha}^{(1)}}\bar{z}^{\bar{\beta}^{(1)}}z^{\bar{\alpha}^{(2)}}\bar{z}^{\bar{\beta}^{(2)}}$ is of (K, μ') -low momentum. The derivative in the Poisson bracket is on I, θ .
3. one has $z^{\alpha^{(1)}}\bar{z}^{\beta^{(1)}} = z^{\bar{\alpha}^{(1)}}\bar{z}^{\bar{\beta}^{(1)}}z_m^\sigma z_n^{\sigma'} z_j^{\sigma_1}$ and $z^{\alpha^{(2)}}\bar{z}^{\beta^{(2)}} = z^{\bar{\alpha}^{(2)}}\bar{z}^{\bar{\beta}^{(2)}}z_j^{-\sigma_1}$ where $\min(|m|, |n|) \geq \lambda'K^\tau$ have a (ℓ, τ_1) cut for some ℓ and $z^{\bar{\alpha}^{(1)}}\bar{z}^{\bar{\beta}^{(1)}}z^{\bar{\alpha}^{(2)}}\bar{z}^{\bar{\beta}^{(2)}}$ is of (K, μ') -low momentum. The derivative in the Poisson bracket is on w_j .
4. one has $z^{\alpha^{(1)}}\bar{z}^{\beta^{(1)}} = z^{\bar{\alpha}^{(1)}}\bar{z}^{\bar{\beta}^{(1)}}z_m^\sigma$ and $z^{\alpha^{(2)}}\bar{z}^{\beta^{(2)}} = z^{\bar{\alpha}^{(2)}}\bar{z}^{\bar{\beta}^{(2)}}z_n^{\sigma'}$ where $\min(|m|, |n|) \geq \lambda'K^\tau$ have a (ℓ, τ_1) cut for some ℓ and $z^{\bar{\alpha}^{(1)}}\bar{z}^{\bar{\beta}^{(1)}}z^{\bar{\alpha}^{(2)}}\bar{z}^{\bar{\beta}^{(2)}}$ is of (K, μ') -low momentum. The derivative in the Poisson bracket is on I, θ .

In case 1. we apply momentum conservation to both monomials and obtain

$$\sigma_1 j = -\sigma m - \pi(k^{(1)}, \bar{\alpha}^{(1)}, \bar{\beta}^{(1)}) = \sigma' n + \pi(k^{(2)}, \bar{\alpha}^{(2)}, \bar{\beta}^{(2)}).$$

Recall that

$$\sum_{l \in \mathbb{Z}_1^d} |l|(\bar{\alpha}_l^{(1)} + \bar{\beta}_l^{(1)} + \bar{\alpha}_l^{(2)} + \bar{\beta}_l^{(2)}) \leq \mu'K^3 \longrightarrow \sum_{l \in \mathbb{Z}_1^d} |l|(\bar{\alpha}_l^{(i)} + \bar{\beta}_l^{(i)}) \leq \mu'K^\tau$$

and by hypothesis $|k^{(i)}| \leq K$, this implies that $|j| > \lambda' K^\tau - \mu' K^3 - CK > \lambda K^\tau$ for $K > \mathcal{K}'$ respecting (4.13) (here C is a constant so that $|\pi(k)| \leq C|k|$). Hence $\min(|m|, |n|, |j|) > \lambda K^\tau$. By momentum conservation $|\sigma m + \sigma_1 j|, |-\sigma_1 j + \sigma' n| \leq CK + \mu' K^3 \leq 5K^3$; by hypothesis n, m have a (ℓ, τ_1) cut. By Lemma 2.3 also $j \xrightarrow{K} [w_i; q_i]$ has a (ℓ, τ_1) cut. Then $e^{i(k^{(i)}, \theta)} z^{\alpha^{(i)}} \bar{z}^{\beta^{(i)}}$ are by definition $(K, \lambda, \mu, \tau_1)$ bilinear. The derivative in the Poisson bracket is on j which is a high momentum variable.

As m, n run over all possible vectors in \mathbb{Z}_1^d with $\min(|m|, |n|) \geq \lambda' K$, we obtain the first term in formula (A.4).

In case 2. following the same argument $e^{i(k^{(1)}, \theta)} z^{\alpha^{(1)}} \bar{z}^{\beta^{(1)}}$ is $(K, \lambda', \mu', \tau_1)$ bilinear and $e^{i(k^{(2)}, \theta)} z^{\alpha^{(2)}} \bar{z}^{\beta^{(2)}}$ is (K, μ') low momentum. We obtain the second contribution in formula (A.4).

In case 3. we apply momentum conservation to the second monomial and obtain $-\sigma_1 j = -\pi(k^{(2)}, \bar{\alpha}^{(2)}, \bar{\beta}^{(2)})$. This implies that

$$|j| + \sum_{l \in \mathbb{Z}_1^d} |l|(\bar{\alpha}_l^{(1)} + \bar{\beta}_l^{(1)}) \leq |\pi(k^{(2)}, \bar{\alpha}^{(2)}, \bar{\beta}^{(2)})| + \sum_{l \in \mathbb{Z}_1^d} |l|(\bar{\alpha}_l^{(1)} + \bar{\beta}_l^{(1)}) \leq$$

$$CK + \sum_{l \in \mathbb{Z}_1^d} |l|(\bar{\alpha}_l^{(1)} + \bar{\beta}_l^{(1)} + \bar{\alpha}_l^{(2)} + \bar{\beta}_l^{(2)}) \leq \mu' K^3 + CK \leq \mu K^3$$

if $K > \mathcal{K}'$ with \mathcal{K}' satisfying (4.13). Then $e^{i(k^{(1)}, \theta)} z^{\alpha^{(1)}} \bar{z}^{\beta^{(1)}}$ is, by definition, $(K, \lambda, \mu, \tau_1)$ bilinear and $e^{i(k^{(2)}, \theta)} z^{\alpha^{(2)}} \bar{z}^{\beta^{(2)}}$ is $(K, 2\mu)$ low momentum. The derivative in the Poisson bracket is on j which is a low momentum variable. We obtain the third contribution in formula (A.4).

In case 4. we apply momentum conservation to both monomials, we get

$$\min(|\sigma m|, |\sigma' n|) \leq \max(|-\pi(k^{(i)}, \bar{\alpha}^{(i)}, \bar{\beta}^{(i)})|) \leq CK + \mu' K^3,$$

which is in contradiction to the hypothesis $\min(|m|, |n|) \geq \lambda' K^\tau$. Hence case 4. does not give any contribution.

The third line in formula (A.4) is dealt just as the second line by exchanging the indexes 1, 2. \blacksquare

In order to show that $\{f^{(1)}, f^{(2)}\}$ is quasi-Töplitz, for all $K > \mathcal{K}'$ and τ_1 we have to provide a decomposition

$$\Pi_{K, \lambda', \mu', \tau_1} \{f^{(1)}, f^{(2)}\} = \mathcal{F}^{(1,2)} + K^{-4d\tau_1} \bar{f}^{(1,2)}$$

so that $\mathcal{F}^{(1,2)} \in \mathbb{F}(K, \tau_1)$ and

$$\|X_{\mathcal{F}^{(1,2)}}\|_{r', s'} \|X_{\bar{f}^{(1,2)}}\|_{r', s'} < \delta^{-1} C \|X_{f^{(1)}}\|_{r, s}^T \|X_{f^{(1)}}\|_{r, s}^T. \quad (\text{A.5})$$

for some constant C .

We substitute in formula (A.4) $\Pi_{K,\lambda',\mu',\tau_1} f^{(i)} = \mathcal{F}^{(i)} + K^{-4d\tau_1} \bar{f}^{(i)}$, with $\mathcal{F}^{(i)} \in \mathbb{F}(\tau_1, K)$.

Lemma A.2 *Consider the function*

$$\mathcal{F}^{(1,2)} = \Pi_{K,\lambda',\mu',\tau_1} \left(\{\mathcal{F}^{(1)}, \mathcal{F}^{(2)}\}^H + \{\mathcal{F}^{(1)}, \Pi_{K,2\mu}^L f^{(2)}\}^{(I,\theta)+L} + \{\Pi_{K,2\mu}^L f^{(1)}, \mathcal{F}^{(2)}\}^{(I,\theta)+L} \right)$$

where we have denoted $\{\cdot, \cdot\}^{(I,\theta)+L} = \{\cdot, \cdot\}^{(I,\theta)} + \{\cdot, \cdot\}^L$. (i) One has $\mathcal{F}^{(1,2)} \in \mathbb{F}(\tau_1, K)$. (ii) Setting $\bar{f}^{(1,2)} = K^{4d\tau_1} (\Pi_{K,\lambda',\mu',\tau_1} \{f^{(1)}, f^{(2)}\} - \mathcal{F}^{(1,2)})$ one has that the bounds (A.5) hold.

Proof: To prove the first statement it is useful to write

$$\mathcal{F}^{(i)} = \sum_{\substack{[v_i; p_i]_\ell \in \mathcal{H}_K \\ |p_\ell| < \mu K^{\tau_1}}} \sum_{\sigma, \sigma' = \pm 1} \sum_{m, n}^* [\mathcal{F}^{(i)}]^{\sigma, \sigma'}(\theta, I, w^L; \sigma m + \sigma' n, [v_i; p_i]_\ell) z_m^\sigma z_n^{\sigma'}$$

where \sum^* is the sum over those n, m which respect (2.11) and have the (ℓ, τ_1) cut $[v_i; p_i]_\ell$ with the parameters λ', μ' . For compactness of notation we will omit the dependence on (θ, I, w^L) .

The fact that $\{\mathcal{F}^{(1)}, \Pi_{K,2\mu}^L f^{(2)}\}^{I,\theta+L} \in \mathbb{F}(\tau_1, K)$ is obvious. Indeed the coefficient of $z_m^\sigma z_n^{\sigma'}$ is

$$\{\mathcal{F}^{(1)}(\sigma m + \sigma' n, [v_i; p_i]_\ell), \Pi_{K,2\mu}^L f^{(2)}\}^{I,\theta+L},$$

the same for $\{\mathcal{F}^{(2)}, \Pi_{K,2\mu}^L f^{(1)}\}^{I,\theta+L}$.

Suppose now that n, m respect (2.11) and have the (ℓ, τ_1) cut $[v_i; p_i]_\ell$ with the parameters λ', μ' . By the rules of Poisson brackets the coefficient of $z_m^\sigma z_n^{\sigma'}$ in the expression $\{\mathcal{F}^{(1)}, \mathcal{F}^{(2)}\}^H$ is

$$\sum_{\substack{r \in \mathbb{Z}_1^d, \sigma_1 = \pm 1 \\ |r| \geq \lambda K^\tau \\ |\sigma m + \sigma_1 r| \leq \mu K^3 \\ |-\sigma_1 r + \sigma' n| \leq \mu K^3}} -\sigma_1 [\mathcal{F}^{(1)}]^{\sigma, \sigma_1}(\sigma m + \sigma_1 r, [v_i; p_i]_\ell) [\mathcal{F}^{(2)}]^{-\sigma_1, \sigma'}(-\sigma_1 r + \sigma' n, [w_i; q_i]_\ell); \quad (\text{A.6})$$

Since $|\sigma m + \sigma_1 r|, |\sigma' n - \sigma_1 r| \leq \mu K^3$ and $|m|, |n| > \lambda' K^\tau$ we have that the condition $|r| > \lambda K^\tau$ is automatically fulfilled. By Lemma 2.3 r, n, m all have a (ℓ, τ_1) cut with parameters (λ, μ) . We set $m \xrightarrow{K} [v_i; p_i]$, $n \xrightarrow{K} [v'_i; p'_i]$, $r \xrightarrow{K} [w_i; q_i]$ and suppose without loss of generality that

$$(p_1, \dots, p_\ell, v_1, \dots, v_\ell) \preceq (q_1, \dots, q_\ell, w_1, \dots, w_\ell) \preceq (p'_1, \dots, p'_\ell, v'_1, \dots, v'_\ell).$$

Again by Lemma 2.3 $\langle v_i \rangle_\ell = \langle v'_i \rangle_\ell = \langle w_i \rangle_\ell$, moreover $[w_i; q_i]_\ell$ is completely fixed by $[v_i; p_i]_\ell$, σ, σ_1 and by $\sigma m + \sigma_1 r := h$. Then we may change variables in the sum over r in (A.6):

$$\sum_{\sigma_1 = \pm 1} \sum_{\substack{h: |h| < \mu K^3 \\ |\sigma m + \sigma'_1 n - h| \leq \mu K^3}} -\sigma_1 [\mathcal{F}^{(1)}]^{\sigma, \sigma_1}(h, [v_i; p_i]_\ell) [\mathcal{F}^{(2)}]^{-\sigma_1, \sigma'_1}(\sigma m + \sigma'_1 n - h; [w_i; q_i]_\ell),$$

this expression only depends on $[v_i; p_i]_\ell$. The estimate (A.5) for $\mathcal{F}^{(1,2)}$ follows by Cauchy estimates since

$$\|X_{\mathcal{F}^{(1,2)}}\|_{r', s'} \leq \|X_{\{\mathcal{F}^{(1)}, \mathcal{F}^{(2)}\}}\|_{r', s'} + \|X_{\{\mathcal{F}^{(1)}, f^{(2)}\}}\|_{r', s'} + \|X_{\{\mathcal{F}^{(2)}, f^{(1)}\}}\|_{r', s'}.$$

We now compute:

$$\begin{aligned} \bar{f} &= \Pi_{K, \lambda', \mu', \tau_1} \left(\{\Pi_{K, \lambda, \mu, \tau_1} f^{(1)}, \bar{f}^{(2)}\}^H + \{\bar{f}^{(1)}, \mathcal{F}^{(2)}\}^H \right. \\ &\quad + \{\bar{f}^{(1)}, \Pi_{K, \mu}^L f^{(2)}\}^{I, \theta} + \{\bar{f}^{(1)}, \Pi_{K, \mu}^L f^{(2)}\}^L + K^{4d\tau_1} \{\Pi_K^U f^{(1)}, f^{(2)}\} \\ &\quad \left. \{\Pi_{K, \mu}^L f^{(1)}, \bar{f}^{(2)}\}^{I, \theta} + \{\Pi_{K, \mu}^L f^{(1)}, \bar{f}^{(2)}\}^L + K^{4d\tau_1} \{f^{(1)}, \Pi_K^U f^{(2)}\} \right). \end{aligned}$$

Since $e^{-K(s-s')} < K^{-\tau}$, one has

$$\|X_{(w^H, F^U w^H)}\|_{r', s'} \leq K^{-\tau} \|X_{f^{(1)}}\|_{r, s} \|X_{f^{(2)}}\|_{r, s},$$

by the smoothing estimates. The estimate (A.5) follows.

Proof: (Proposition 4) Proposition 4(i) follows from the previous Lemma.

(ii) Given $f^{(i)}$, $i = 1, \dots, N$ as in item (i), and applying repeatedly (4.13), the nested Poisson bracket

$$\{f^{(1)}, \{f^{(2)}, \dots, \{f^{(N-1)}, f^{(N)}\} \dots\}$$

is quasi-Töplitz in $\mathcal{D}(r_+, s_+)$ with parameters $(\mathcal{K}_+, \lambda_+, \mu_+)$ if

$$\mu_+ K^3 + 2NK < \mu K^3, \quad 2N\mu_+ K^3 < (\lambda_+ - \lambda) K^{4d\tau_0 - 1}, \quad (\text{A.7})$$

for all $K > \mathcal{K}_+$

For given K we bound all the terms in $e^{\{F, \cdot\}} G$ containing $N > K$ Poisson brackets by $K^{-\tau}$ by using the analyticity of the nested Poisson bracket in $\mathcal{D}(r_+, s_+)$ and the denominator $N!$. We then apply (A.7) with $N = K$, we get the restriction (4.15). \blacksquare

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